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CERTAIN PROBLEMS OF CLOSEST APPROXIMATION*

BY DUNHAM JACKSON

1. *Introduction.* In connection with the theory of systems of polynomials which are orthogonal and normalized with respect to a given weight function an important question is that of the order of magnitude of the nth polynomial of the sequence as n becomes infinite. In the fundamentally important case of Jacobi polynomials† as well as for the systems of polynomials corresponding to much more general weight functions‡ asymptotic formulas show that the polynomials of the normalized sequence remain bounded, at least in the interior of the interval for which they are constructed. This paper is in part devoted to a much less profound but considerably broader study of upper bounds for the order of magnitude of the polynomials under still more general hypotheses as to the character of the weight function. It is believed to be of interest by reason of the simplicity of the methods employed, and their ready applicability to the obtaining of results with regard to the behavior of the polynomials even at points where the weight function vanishes.

Attention is given also to similar problems in the case of trigonometric sums, which are in some respects more readily accessible to treatment than polynomials.

The rest of the paper is concerned with the convergence of the development of a given function in series of the orthogonal polynomials or trigonometric sums, or more directly, as the terms of the series do not enter explicitly into the calculation, with

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* Presented before the Society and Section A of the American Association for the Advancement of Science, by invitation, June 21, 1933, on the occasion of the address by Professor L. Fejér.

† See G. Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres, et sur une classe étendue de développements en série*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56.

‡ See G. Szegö, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Mathematische Annalen, vol. 86 (1922), pp. 114–139; S. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini*, Journal de Mathématiques, (9), vol. 9 (1930), pp. 127–177, and vol. 10 (1931), pp. 219–286.

the convergence of the corresponding processes of least-square approximation. More generally, an arbitrary positive power of the error is admitted in place of the square in the criterion of closest approximation. The author has pointed out on various occasions the usefulness of Bernstein's theorem in connection with the study of problems of this sort.* Other writers have made effective use of Hölder's inequality in similar situations.† The present treatment is characterized as to method by the use of Bernstein's theorem, or Markoff's theorem, and Hölder's inequality in succession, and as to results by the comparative liberality of the hypotheses with regard to vanishing of the weight function.

2. *Normalized Trigonometric Sums.* Detailed consideration will be given first, for the sake of simplicity, not to polynomials but to trigonometric sums.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order,‡ and s an arbitrary positive number, and let

$$H_{ns} = \int_{-\pi}^{\pi} |T_n(x)|^s dx.$$

Let μ_n be the maximum of $|T_n(x)|$, and let x_0 be a value of x for which $|T_n(x_0)| = \mu_n$. By Bernstein's theorem,

$$|T_n'(x)| \leqq n\mu_n$$

everywhere. For $|x - x_0| \leqq 1/(2n)$, by the law of the mean,

$$|T_n(x) - T_n(x_0)| \leqq \mu_n/2,$$

and

$$|T_n(x)| \geqq \mu_n/2.$$

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* See, for example, D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI (hereafter referred to as Colloquium), 1930, Chapter III.

† See, for example, J. Shohat, *On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval*, Mathematische Annalen, vol. 102 (1930), pp. 157–175; J. L. Walsh, *On the overconvergence of sequences of polynomials of best approximation*, Transactions of this Society, vol. 32 (1930), pp. 794–816; W. H. McEwen, *Problems of closest approximation connected with the solution of linear differential equations*, Transactions of this Society, vol. 33 (1931), pp. 979–997.

‡ The words "of the nth order" or "of the nth degree" will be understood throughout to mean of the nth order or of the nth degree *at most*.

As the last relation holds throughout an interval of length $1/n$ at least, it is certain that

$$H_{ns} \geqq (1/n)(\mu_n/2)^s,$$

whence

$$\mu_n \leqq 2(nH_{ns})^{1/s}.$$

The conclusion is as follows.

LEMMA 1. *If $T_n(x)$ is a trigonometric sum of the nth order, if*

$$H_{ns} = \int_{-\pi}^{\pi} |T_n(x)|^s dx,$$

and if μ_n is the maximum of $|T_n(x)|$, then

$$\mu_n \leqq 2(nH_{ns})^{1/s}.$$

This statement may be supplemented by an immediate corollary.

COROLLARY. *If sums $T_n(x)$, each of order indicated by its subscript, are defined for an infinite succession of values of n so that H_{ns} is bounded, for a fixed value of s, then $|T_n(x)|$ has an upper bound of the order of $n^{1/s}$.*

Suppose now that $\rho(x)$ is a summable function of period 2π having a positive lower bound:

$$\rho(x) \geqq v > 0$$

for all values of x. Let $T_n(x)$ be a trigonometric sum of the nth order such that

$$\int_{-\pi}^{\pi} \rho(x)[T_n(x)]^2 dx = 1. \tag{1}$$

Inasmuch as $[\rho(x)]^{-1} \leqq 1/v$,

$$H_{n2} = \int_{-\pi}^{\pi} [T_n(x)]^2 dx \leqq 1/v,$$

and application of the Lemma, through its Corollary, with $s=2$, yields the following result.*

* See also D. Jackson, *Orthogonal trigonometric sums*, presently to be published in the Annals of Mathematics. It is to be noted that the proof does not require that the sums $T_n(x)$ form an orthogonal system.

THEOREM 1. *If $\rho(x)$ is a summable function having a positive lower bound, and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that* (1) *is satisfied, then $|T_n(x)|$ has an upper bound of the order of $n^{1/2}$.*

Let the hypothesis with regard to a positive lower bound of $\rho(x)$ be replaced by the less restrictive assumption that $\rho(x)$ is nowhere negative and that $[\rho(x)]^{-r}$ is summable over a period, for some positive (not necessarily integral) value of r. Let $T_n(x)$ again be a trigonometric sum of the nth order satisfying (1). Let

$$s = 2r/(r+1), \qquad p = 2/s = 1 + (1/r) > 1.$$

In the integral

$$H_{ns} = \int_{-\pi}^{\pi} |T_n(x)|^s dx,$$

let the integrand be regarded as the product of the factors $[\rho(x)]^{-1/p}$ and $[\rho(x)]^{1/p}|T_n(x)|^s$. Then, by Hölder's inequality,

$$H_{ns} \leqq \left[\int_{-\pi}^{\pi} \{[\rho(x)]^{-1/p}\}^{p/(p-1)} dx\right]^{(p-1)/p} \cdot \left[\int_{-\pi}^{\pi} \{[\rho(x)]^{1/p} |T_n(x)|^s\}^p dx\right]^{1/p}$$

$$= \left[\int_{-\pi}^{\pi} [\rho(x)]^{-r} dx\right]^{1/(r+1)} \left[\int_{-\pi}^{\pi} \rho(x)[T_n(x)]^2 dx\right]^{1/p}.$$

The first integral in brackets on the right, which exists by hypothesis, is independent of n, and the value of the other integral is 1, by the hypothesis on $T_n(x)$. So H_{ns} is bounded, with $s=2r/(r+1)$, and the Corollary of the Lemma goes over into the following form.*

THEOREM 2. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable, $r>0$, and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that* (1) *is satisfied, then $|T_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$.*

* See also D. Jackson, Annals of Mathematics, loc. cit., for the particular case $r=1$.

3. *Normalized Polynomials, Upper Bounds for Entire Interval.* Let $P_n(x)$ be a polynomial of the nth degree, and μ_n the maximum of $|P_n(x)|$ for $-1 \leqq x \leqq 1$. According to Markoff's theorem

$$|P_n'(x)| \leqq n^2\mu_n$$

for $-1 \leqq x \leqq 1$. Let x_0 be a point of the interval $(-1, 1)$ at which $|P_n(x)| = \mu_n$. If x is a point of $(-1, 1)$ distant from x_0 by not more than $1/(2n^2)$,

$$|P_n(x_0) - P_n(x)| \leqq \mu_n/2, \qquad |P_n(x)| \geqq \mu_n/2.$$

At least one of the intervals

$$x_0 - 1/(2n^2) \leqq x \leqq x_0, \qquad x_0 \leqq x \leqq x_0 + 1/(2n^2),$$

is wholly contained in $(-1, 1)$, and $|P_n(x)| \geqq \mu_n/2$ consequently throughout an interval of length at least $1/(2n^2)$. Hence, if $s > 0$ and

$$H_{ns} = \int_{-1}^{1} |P_n(x)|^s dx,$$

it is certain that

$$H_{ns} \geqq \frac{1}{2n^2}\left(\frac{\mu_n}{2}\right)^s, \quad \mu_n \leqq 2(2n^2H_{ns})^{1/s}.$$

More generally, let

$$H_{ns} = \int_a^b |P_n(x)|^s dx$$

for an arbitrary interval (a, b), and let μ_n be the maximum of $|P_n(x)|$ for $a \leqq x \leqq b$. By virtue of the transformation $y = (2x - a - b)/(b - a)$, $P_n(x)$ is a polynomial of the nth degree in y, $Q_n(y)$, having μ_n as the maximum of its absolute value for $-1 \leqq y \leqq 1$, and

$$\int_{-1}^{1} |Q_n(y)|^s dy = \frac{2}{b-a}\int_a^b |P_n(x)|^s dx = \frac{2H_{ns}}{b-a}.$$

Application of the preceding paragraph to $Q_n(y)$ gives an upper bound for μ_n.

LEMMA 2. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H_{ns} = \int_a^b |P_n(x)|^s dx,$$

and if μ_n *is the maximum of* $|P_n(x)|$ *for* $a \leqq x \leqq b$, *then*

$$\mu_n \leqq 2[4n^2 H_{ns}/(b-a)]^{1/s}.$$

If H_{ns} *is bounded for a succession of polynomials* $P_n(x)$, *with a fixed value of* s, $|P_n(x)|$ *has an upper bound of the order of* $n^{2/s}$ *for* $a \leqq x \leqq b$.

An immediate consequence for $s=2$, after the analogy of Theorem 1, a result which is well known on the basis of other lines of reasoning, is as follows.

THEOREM 3. *If* $\rho(x)$ *is a summable function having a positive lower bound for* $a \leqq x \leqq b$, *and if polynomials* $P_n(x)$ *of the* n*th degree are constructed for successive values of* n *so that*

$$\int_a^b \rho(x)[P_n(x)]^2 dx = 1, \tag{2}$$

then $|P_n(x)|$ *has an upper bound of the order of* n *for* $a \leqq x \leqq b$.

An obvious generalization may be stated as follows.

COROLLARY. *If* $\rho(x)$ *in Theorem 3, non-negative throughout* (a, b), *is assumed to have a positive lower bound merely for* $c \leqq x \leqq d$, *where* $a \leqq c < d \leqq b$, *without being restricted as to its vanishing outside* (c, d), *then* $|P_n(x)|$ *has an upper bound of the order of* n *for* $c \leqq x \leqq d$.

For

$$\int_c^d \rho(x)[P_n(x)]^2 dx \leqq \int_a^b \rho(x)[P_n(x)]^2 dx = 1,$$

and the proof of the theorem can be applied directly to the interval (c, d).

The reasoning with Hölder's inequality by means of which Theorem 2 was proved leads now to the following theorem.

THEOREM 4. *If* $\rho(x)$ *is a non-negative summable function such that* $[\rho(x)]^{-r}$ *is summable over* (a, b), *with* $r>0$, *and if polynomials* $P_n(x)$ *of the* n*th degree are constructed for successive values of* n *so that* (2) *is satisfied, then* $|P_n(x)|$ *has an upper bound of the order of* $n^{(r+1)/r}$ *for* $a \leqq x \leqq b$.

The proof consists in showing, by adaptation of the formulas

that we have previously employed, that H_{ns} again is bounded with $s=2r/(r+1)$.

To the result just formulated we may add the following statement.

COROLLARY. *If $\rho(x)$ in Theorem 4 is non-negative and summable over (a, b), and if $[\rho(x)]^{-r}$ is assumed merely to be summable over (c, d), where $a \leqq c < d \leqq b$, then $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/r}$ for $c \leqq x \leqq d$.*

4. *Normalized Polynomials, Upper Bounds for Entire Interval by Trigonometric Substitution.* Let $P_n(x)$ again be a polynomial of the nth degree, and μ_n the maximum of its absolute value for $-1 \leqq x \leqq 1$. Let

$$H'_{ns} = \int_{-1}^{1} (1 - x^2)^{-1/2} |P_n(x)|^s dx,$$

still with the understanding that $s>0$. By the substitution $x=\cos\theta$ the integral becomes

$$H'_{ns} = \int_0^{\pi} |P_n(\cos\theta)|^s d\theta = \frac{1}{2}\int_{-\pi}^{\pi} |P_n(\cos\theta)|^s d\theta.$$

As $P_n(\cos\theta)$ is a trigonometric sum of the nth order in θ, Lemma 1 is applicable, with the conclusion that $\mu_n \leqq 2(2nH'_{ns})^{1/s}$.

For a general interval, let

$$H'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2} |P_n(x)|^s dx.$$

If $y=(2x-a-b)/(b-a)$ and $P_n(x)=Q_n(y)$,

$$\int_{-1}^{1} (1 - y^2)^{-1/2} |Q_n(y)|^s dy = H'_{ns}.$$

Application of the preceding paragraph to this integral gives the following lemma.

LEMMA 3. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2} |P_n(x)|^s dx,$$

and if μ_n is the maximum of $|P_n(x)|$ for $a \leqq x \leqq b$, then

$$\mu_n \leqq 2(2nH'_{ns})^{1/s}.$$

If H_{ns}' is bounded for a succession of polynomials $P_n(x)$, with fixed s, $|P_n(x)|$ has an upper bound of the order of $n^{1/s}$ for $a \leqq x \leqq b$.

For $s=2$ this yields the following result.

THEOREM 5. *If $\rho(x)$ is a summable function such that $\rho(x)[(b-x)(x-a)]^{1/2}$ has a positive lower bound over (a, b), and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, then $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ for $a \leqq x \leqq b$.*

COROLLARY. *If $\rho(x)$ is non-negative and summable over (a, b), and if $\rho(x)[(d-x)(x-c)]^{1/2}$ has a positive lower bound over (c, d), where $a \leqq c < d \leqq b$, then $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ for $c \leqq x \leqq d$.*

Hölder's inequality is to be applied this time to the integral of the product of the factors

$$[\rho(x)]^{-1/p}[(b-x)(x-a)]^{-1/2},\ [\rho(x)]^{1/p}\,|P_n(x)|^s,$$

where s and p are related to r as before, to obtain the following theorem.

THEOREM 6. *If $\rho(x)$ is a non-negative summable function such that*

$$[\rho(x)]^{-r}[(b-x)(x-a)]^{-(r+1)/2}$$

is summable over (a, b), with $r>0$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, then $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ for $a \leqq x \leqq b$.

It is perhaps not necessary to state the corollary relating to an interval (c, d) contained in (a, b).

5. *Normalized Polynomials, Upper Bounds for Interior of Interval.* The trigonometric substitution of §4 is useful also in connection with the integral denoted by H_{ns} in §3. Let

$$H_{ns} = \int_{-1}^{1} |P_n(x)|^s dx$$

once more, and let $x=\cos\theta$. Then

$$H_{ns} = \int_0^{\pi} \sin\theta\, |P_n(\cos\theta)|^s d\theta.$$

Let N be the smallest integer satisfying the condition that $N \geqq 1/s$. This relation means that $1 \leqq Ns$, so that $\sin\theta \geqq (\sin\theta)^{Ns}$ for $0 \leqq \theta \leqq \pi$, and

$$H_{ns} \geqq \int_0^\pi |\sin^N\theta\, P_n(\cos\theta)|^s d\theta = \frac{1}{2}\int_{-\pi}^{\pi} |\sin^N\theta\, P_n(\cos\theta)|^s d\theta.$$

Let μ_n' be the maximum of $|\sin^N\theta\, P_n(\cos\theta)|$. As the expression in bars is a trigonometric sum of order $n+N$, it follows from Lemma 1 that

$$\mu_n' \leqq 2[2(n+N)H_{ns}]^{1/s}.$$

Hence

$$|P_n(\cos\theta)| \leqq \frac{2[2(n+N)H_{ns}]^{1/s}}{|\sin^N\theta|},$$

or, in terms of the original variable x,

$$|P_n(x)| \leqq \frac{2[2(n+N)H_{ns}]^{1/s}}{(1-x^2)^{N/2}},$$

for $-1<x<1$. With the aid of a further change of variable the conclusion may be expressed in the following form.

LEMMA 4. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H_{ns} = \int_a^b |P_n(x)|^s dx,$$

and if N is the smallest integer $\geqq 1/s$, then

$$|P_n(x)| \leqq \frac{K[(n+N)H_{ns}]^{1/s}}{[(b-x)(x-a)]^{N/2}}$$

for $a<x<b$, where

$$K = 2\cdot 2^{1/s}[(b-a)/2]^{N-(1/s)}.$$

The significance of this result is that for a fixed x interior to (a, b), or for any closed interval interior to (a, b), the upper bound obtained is of the order of $(nH_{ns})^{1/s}$ for fixed s, as compared with $(n^2 H_{ns})^{1/s}$ in Lemma 2 or $(nH_{ns}')^{1/s}$ in Lemma 3; the earlier lemmas, on the other hand, apply to the entire closed interval $a \leqq x \leqq b$.

It may be noted, though perhaps as a point of minor interest, that the statement of Lemma 4 becomes somewhat simpler and more compact if s is the reciprocal of an integer, so that $N=1/s$.

Application of the Lemma is to be made through the following corollary.

COROLLARY. *If H_{ns} is bounded for a succession of polynomials $P_n(x)$, with a fixed value of s, then $|P_n(x)|$ has an upper bound of the order of $n^{1/s}$ throughout any closed interval interior to (a, b).*

For $s=2$ there is obtained the following well known supplement to Theorem 3.

THEOREM 7. *If $\rho(x)$ is a summable function having a positive lower bound for $a \leqq x \leqq b$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any closed interval interior to (a, b).*

If $\rho(x)$, non-negative throughout (a, b), has a positive lower bound in an interval (c, d) contained in (a, b), $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any closed interval interior to (c, d).

A similar continuation of Theorem 4 requires no new calculation; it is a question merely of combining Lemma 4, through its Corollary, with the observation already made as to the boundedness of H_{ns} for $s=2r/(r+1)$.

THEOREM 8. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable over (a, b), with $r>0$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ throughout any closed interval interior to (a, b).*

This also can be adapted to the hypothesis that $[\rho(x)]^{-r}$ is summable merely over an interval (c, d) contained in (a, b).

6. *Convergence of Trigonometric Approximation.* A schedule of propositions corresponding to those listed above can be worked out with reference to the convergence of trigonometric or polynomial approximations determined by the minimizing of an integral containing a power of the error. In the case of least squares, as is well known, the approximating functions can be regarded as partial sums of developments in series of polynomials or trigonometric sums orthogonal with respect to the weight function in question. The terms of the series will not be explicitly in evidence, however; the treatment of convergence, for least squares as well as in the case of other powers, will be based directly on consideration of the magnitude of the integral which is minimized.

As in the earlier part of the paper, attention will first be given to trigonometric sums.

Let $f(x)$ be a given continuous function of period 2π, and let $T_n(x)$ and $t_n(x)$ be arbitrary trigonometric sums of the nth order. For a given positive s, let

$$G_{ns} = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^s dx.$$

Let the difference $f(x)-t_n(x)$ be denoted by $r_n(x)$, and let ϵ_n be an upper bound for $|r_n(x)|$:

$$|f(x) - t_n(x)| \leqq \epsilon_n.$$

Let $T_n(x)-t_n(x)=\tau_n(x)$, so that $f(x)-T_n(x)\equiv r_n(x)-\tau_n(x)$. Let μ_n be the maximum value of $|\tau_n(x)|$, taken on for $x=x_0$. For $|x-x_0| \leqq 1/(2n)$, inasmuch as $|\tau_n'(x)| \leqq n\mu_n$, by Bernstein's theorem, $|\tau_n(x)|$ remains greater than or equal to $\mu_n/2$. If $\mu_n \geqq 4\epsilon_n$, so that $|r_n(x)| \leqq \mu_n/4$, then

$$|r_n(x) - \tau_n(x)| \geqq \mu_n/4$$

throughout the specified interval of length $1/n$, and

$$G_{ns} = \int_{-\pi}^{\pi} |r_n(x) - \tau_n(x)|^s dx \geqq \frac{1}{n}\left(\frac{\mu_n}{4}\right)^s,$$

from which it follows that

$$\mu_n \leqq 4(nG_{ns})^{1/s}.$$

If the condition $\mu_n \geqq 4\epsilon_n$ is not satisfied, this fact of itself gives an upper bound for μ_n. In any case μ_n has one or the other of the numbers $4\epsilon_n$, $4(nG_{ns})^{1/s}$ for an upper bound, and can not exceed their sum:

$$\mu_n \leqq 4(nG_{ns})^{1/s} + 4\epsilon_n.$$

Since $|r_n(x)| \leqq \epsilon_n$,

$$|f(x) - T_n(x)| = |r_n(x) - \tau_n(x)| \leqq \epsilon_n + \mu_n \leqq 4(nG_{ns})^{1/s} + 5\epsilon_n.$$

The conclusion may be expressed in the following form:*

LEMMA 5. *If $f(x)$ is a continuous function of period 2π, $T_n(x)$ a trigonometric sum of the nth order, and*

* See Colloquium, pp. 84, 87–88.

$$G_{ns} = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^s dx,$$

and if there exists a trigonometric sum $t_n(x)$ of the nth order such that

$$|f(x) - t_n(x)| \leqq \epsilon_n$$

everywhere, then

$$|f(x) - T_n(x)| \leqq 4(nG_{ns})^{1/s} + 5\epsilon_n$$

for all values of x.

Let $\rho(x)$ be a summable function of period 2π having a positive lower bound, $\rho(x) \geqq v > 0$, and let $T_n(x)$ be determined among all trigonometric sums of the nth order as one for which the integral

$$\int_{-\pi}^{\pi} \rho(x) |f(x) - T_n(x)|^m dx \tag{3}$$

has its minimum value, the exponent m being a given positive number. The question of the existence and uniqueness or non-uniqueness of the minimizing sums, for both trigonometric and polynomial approximation, has been treated extensively elsewhere,* and will not be discussed here. Let the minimum value of the integral be denoted by γ_n. In consequence of the hypothesis on $\rho(x)$,

$$G_{nm} = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^m dx \leqq \gamma_n / v,$$

and by application of Lemma 5

$$|f(x) - T_n(x)| \leqq 4(n\gamma_n/v)^{1/m} + 5\epsilon_n,$$

if $f(x)$ can be approximated by a trigonometric sum $t_n(x)$ of the nth order with a maximum error not exceeding ϵ_n.

* See, for example, D. Jackson, *Note on an ambiguous case of approximation*, Transactions of this Society, vol. 25 (1923), pp. 333–337; *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), pp. 133–154; *Note on the convergence of a sequence of approximating polynomials*, this Bulletin, vol. 37 (1931), pp. 69–72; and other passages referred to in these papers.

Under the same assumption as to the existence of an approximating sum $t_n(x)$ an upper bound can be assigned for γ_n. If the integral of $\rho(x)$ over a period is denoted by R, then by virtue of the minimizing property of $T_n(x)$

$$\gamma_n = \int_{-\pi}^{\pi} \rho(x) \left| f(x) - T_n(x) \right|^m dx$$

$$\leqq \int_{-\pi}^{\pi} \rho(x) \left| f(x) - t_n(x) \right|^m dx \leqq R\epsilon_n^m.$$

With the incidental observation that $5\epsilon_n \leqq 5n^{1/m}\epsilon_n$, it thus becomes apparent that

$$\left| f(x) - T_n(x) \right| \leqq Cn^{1/m}\epsilon_n,$$

where C is independent of x and independent of n.

The minimizing sums $T_n(x)$ will converge uniformly toward the function $f(x)$ as n becomes infinite if sums $t_n(x)$ exist so that $\lim_{n\to\infty} n^{1/m}\epsilon_n = 0$. A well known sufficient condition,* when $m > 1$, is that $f(x)$ have a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0} \omega(\delta)/\delta^{1/m} = 0$. In this theorem and succeeding theorems, however, the formal statement will be in terms of the order of magnitude of ϵ_n, and reference will be made to systematic presentations of the theory of approximation by polynomials and trigonometric sums† for relations between the values attainable for ϵ_n and properties of continuity of $f(x)$.

THEOREM 9. *If $\rho(x)$ is a summable function having a positive lower bound, if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n to minimize the integral* (3), *and if there exist trigonometric sums $t_n(x)$, likewise of the nth order, so that*

$$\left| f(x) - t_n(x) \right| \leqq \epsilon_n,$$

there is a constant C, independent of x and n, such that

$$\left| f(x) - T_n(x) \right| \leqq Cn^{1/m}\epsilon_n.$$

* See Colloquium, p. 89, where a more general form of convergence theorem is given, applying to other sums than those which minimize the error integral. The more general form of statement could be carried through the present paper, but the specific statements with regard to minimizing sums are preferred for the sake of simplicity.

† For example, Colloquium, Chapter I.

The minimizing sums $T_n(x)$ will converge uniformly toward $f(x)$ if sums $t_n(x)$ exist for which $\lim_{n\to\infty} n^{1/m}\epsilon_n = 0$.

A new theorem is obtained if it is assumed merely that $\rho(x)$, in addition to being non-negative and summable, is such that $[\rho(x)]^{-r}$ is summable over a period, for a positive value of r. Let

$$s = mr/(r+1), \qquad p = m/s = 1 + (1/r) > 1,$$

and let Hölder's inequality be applied to the integral of the product of the factors

$$[\rho(x)]^{-1/p}, \qquad [\rho(x)]^{1/p}\,|f(x) - T_n(x)|^s,$$

to give a comparison between this integral and a product of integrals involving these factors with the exponents $p/(p-1)$ and p, respectively. The resulting inequality is

$$G_{ns} = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^s dx$$

$$\leqq \left[\int_{-\pi}^{\pi} [\rho(x)]^{-r} dx\right]^{1/(r+1)} \cdot \left[\int_{-\pi}^{\pi} \rho(x)\,|f(x) - T_n(x)|^m dx\right]^{r/(r+1)}.$$

If γ_n and ϵ_n are used with the same meanings as before, the last member does not exceed a constant multiple of $\gamma_n^{r/(r+1)}$ or of $\epsilon_n^{mr/(r+1)} = \epsilon_n^s$, and Lemma 5 gives for $|f(x) - T_n(x)|$ an upper bound of the order of $n^{1/s}\epsilon_n$.

THEOREM 10. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable, $r>0$, if $T_n(x)$ for each n is a trigonometric sum of the nth order minimizing (3), and if sums $t_n(x)$ exist so that*

$$|f(x) - t_n(x)| \leqq \epsilon_n,$$

there is a constant C, independent of x and n, such that

$$|f(x) - T_n(x)| \leqq Cn^{(r+1)/(mr)}\epsilon_n.$$

The sums $T_n(x)$ will converge uniformly toward $f(x)$ if sums $t_n(x)$ exist for which

$$\lim_{n\to\infty} n^{(r+1)/(mr)}\epsilon_n = 0.$$

The last condition will be satisfied if $s=mr/(r+1)>1$ and the function $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0}\omega(\delta)/\delta^{1/s}=0$, or if $m>1$, $r=1/(m-1)$, and $f(x)$ has a continuous derivative. When $s<1$ a corresponding statement would involve further hypotheses on $f'(x)$, or, on occasion, higher derivatives of $f(x)$.

It is to be noted that Theorem 10, like the other even-numbered theorems throughout the paper, *applies even at points where* $\rho(x)$ *vanishes.*

7. *Convergence of Polynomial Approximation over Entire Interval.* The theorems that are still to be enumerated can be proved by combination of devices that have been employed above, and may be dismissed with a brief summary.

Throughout the remainder of the paper it will be understood that $f(x)$ is a given function continuous for $a\leqq x\leqq b$, and that $\rho(x)$ is non-negative and summable over (a, b). Whenever the symbol ϵ_n is used, it is with the implication that there exists a polynomial $p_n(x)$, of the nth degree, such that

$$|f(x)-p_n(x)|\leqq\epsilon_n$$

for $a\leqq x\leqq b$.

A proof related to that of Lemma 5 as the proof of Lemma 2 is related to that of Lemma 1 gives the following result.*

LEMMA 6. *If* $P_n(x)$ *is a polynomial of the nth degree, and*

$$G_{ns}=\int_a^b|f(x)-P_n(x)|^s dx,$$

then

$$|f(x)-P_n(x)|\leqq 4[4n^2G_{ns}/(b-a)]^{1/s}+5\epsilon_n$$

for $a\leqq x\leqq b$.

In deriving the following theorems, if the interval (c, d) is not the whole of (a, b), the Lemma is to be restated with reference to the interval (c, d) for the purposes of the demonstration:

THEOREM 11.† *If* $P_n(x)$ *is a polynomial of the nth degree minimizing the integral*

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* See Colloquium, p. 97.

† See Colloquium, p. 98.

$$\int_a^b \rho(x)\,|f(x) - P_n(x)|^m dx, \qquad (m > 0), \tag{4}$$

and if $\rho(x)$ *has a positive lower bound for* $c \leqq x \leqq d$, *where* $a \leqq c < d \leqq b$, *there is a constant* C, *independent of* x *and* n, *such that*

$$|f(x) - P_n(x)| \leqq Cn^{2/m}\epsilon_n$$

for $c \leqq x \leqq d$.

THEOREM 12. *If* $P_n(x)$ *is a polynomial of the* n*th degree minimizing the integral* (4), *and if* $[\rho(x)]^{-r}$ *is summable over* (c, d), *with* $r > 0$, *where* $a \leqq c < d \leqq b$, *there is a constant* C, *independent of* x *and* n, *such that*

$$|f(x) - P_n(x)| \leqq Cn^{2(r+1)/(mr)}\epsilon_n$$

for $c \leqq x \leqq d$.

A sufficient condition for convergence, in terms of the order of magnitude of ϵ_n, is obvious in each case.

8. *Convergence of Polynomial Approximation over Entire Interval by Trigonometric Substitution.* The substitution $x = \cos\theta$ can be employed as in the proof of Lemma 3 to express an integral over the interval $(-1, 1)$ in terms of periodic functions; if $f(x)$ can be approximately represented by a polynomial $p_n(x)$ for $-1 \leqq x \leqq 1$ with an error not exceeding ϵ_n, the even periodic function $f(\cos\theta)$ is represented by the trigonometric sum $p_n(\cos\theta)$ so that

$$|f(\cos\theta) - p_n(\cos\theta)| \leqq \epsilon_n$$

for all values of θ. The further use of a linear substitution relates the interval $(-1, 1)$ to an arbitrary interval (a, b). The resulting transformation of Lemma 5 reads as follows.

LEMMA 7. *If* $P_n(x)$ *is a polynomial of the* n*th degree, and*

$$G'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2}\,|f(x) - P_n(x)|^s dx,$$

then

$$|f(x) - P_n(x)| \leqq 4(2nG'_{ns})^{1/s} + 5\epsilon_n$$

for $a \leqq x \leqq b$.

From this Lemma, restated for an interval (c, d), may be deduced theorems with regard to polynomials of closest approximation.

THEOREM 13. *If $P_n(x)$ is a polynomial of the nth degree minimizing the integral* (4), *and if $\rho(x)[(d-x)(x-c)]^{1/2}$ has a positive lower bound over (c, d), where $a \leqq c < d \leqq b$, there is a constant C, independent of x and n, such that*

$$| f(x) - P_n(x) | \leqq Cn^{1/m}\epsilon_n$$

for $c \leqq x \leqq d$.

THEOREM 14. *If $P_n(x)$ is a polynomial of the nth degree minimizing the integral* (4), *and if*

$$[\rho(x)]^{-r}[(d-x)(x-c)]^{-(r+1)/2}$$

is summable over (c, d), with $r > 0$ where $a \leqq c < d \leqq b$, there is a constant C, independent of x and n, such that

$$| f(x) - P_n(x) | \leqq Cn^{(r+1)/(mr)}\epsilon_n$$

for $c \leqq x \leqq d$.

9. *Convergence of Polynomial Approximation over Interior of Interval.* Adaptation of the reasoning by which Lemma 4 was established leads to the following lemma.

LEMMA 8. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$G_{ns} = \int_a^b | f(x) - P_n(x) |^s dx,$$

and if N is the smallest integer $\geqq 1/s$, then

$$| f(x) - P_n(x) | \leqq \frac{K_1[(n+N)G_{ns}]^{1/s} + K_2\epsilon_n}{[(b-x)(x-a)]^{N/2}}$$

for $a < x < b$, where

$$K_1 = 4 \cdot 2^{1/s}[(b-a)/2]^{N-(1/s)}, \quad K_2 = 5[(b-a)/2]^N.$$

It should perhaps be noted explicitly, as an item in the proof, that if there is a polynomial $p_n(x)$ of the nth degree such that

$|f(x)-p_n(x)| \leqq \epsilon_n$ for $-1 \leqq x \leqq 1$, then $\sin^N \theta\, p_n(\cos\theta)$ is a trigonometric sum of order $n+N$ approximating $\sin^N \theta\, f(\cos\theta)$ so that

$$|\sin^N\theta\, f(\cos\theta) - \sin^N\theta\, p_n(\cos\theta)| \leqq \epsilon_n$$

for all values of θ.

The following theorem, obtained without the use of Hölder's inequality, is a material improvement over a corresponding result previously published.*

THEOREM 15. *If $P_n(x)$ is a polynomial of the nth degree minimizing (4), and if $\rho(x)$ has a positive lower bound for $c \leqq x \leqq d$, where $a \leqq c < d \leqq b$, then $|f(x)-P_n(x)|$ has an upper bound of the order of $n^{1/m}\,\epsilon_n$ throughout any closed interval interior to (c, d).*

A sufficient condition for convergence in the interior of the interval (c, d), when $m>1$, is that $f(x)$ have a modulus of continuity $\omega(\delta)$ throughout (a, b) such that $\lim_{\delta\to 0}\omega(\delta)/\delta^{1/m}=0$; when $m=1$ it is sufficient that $f(x)$ have a continuous derivative throughout (a, b).

Finally, an application of Hölder's inequality already needed for the proof of Theorem 12 gives the following result.

THEOREM 16. *If $P_n(x)$ is a polynomial of the nth degree minimizing (4), and if $[\rho(x)]^{-r}$ is summable over (c, d), with $r>0$, where $a \leqq c < d \leqq b$, then $|f(x)-P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(mr)}\,\epsilon_n$ throughout any closed interval interior to (c, d).*

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* See Colloquium, p. 101.

ON THE USE OF CESÀRO MEANS IN DETERMINING CRITERIA FOR FOURIER CONSTANTS*

BY C. N. MOORE

1. *Introduction.* In the portion of Professor Fejér's address dealing with Fourier series, the positiveness of certain linear operations played a central role. These linear operations were those that arise in forming for a series the partial Cesàro means of various integral orders.

The positiveness of the linear functional operation in the case of a wide class of Fourier series resulted from the positiveness in the case of certain very simple trigonometric series. One such series is the series

$$(1) \qquad \frac{1}{2} + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta + \cdots,$$

whose behavior is of fundamental importance in studying the convergence or summability of the general Fourier series. For the series (1) the partial Cesàro mean of the first order takes the form

$$(2) \qquad \frac{1}{2n} \cdot \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}, \qquad (\theta \neq 2m\pi),$$

and the value $n/2$ for the excepted values of θ, thus being obviously non-negative for all values of θ.

For many questions involving Fourier series further light is shed on the behavior of the series by considering the Cesàro means of non-integral order introduced by Knopp, M. Riesz, and Chapman. These means may be defined as follows. We set

$$(3) \qquad A_0^{(k)} = 1, A_n^{(k)} = \frac{(k+1)\cdots(k+n)}{n!}, \quad (n = 1, 2, \cdots),$$

so that

$$(1 - z)^{-(k+1)} = \sum_{n=0}^{\infty} A_n^{(k)} z^n, \qquad (|z| < 1),$$

* Presented to the Society and Section A of the American Association for the Advancement of Science, by invitation, June 21, 1933, on the occasion of an address by Professor L. Fejér.

the A's thus being the coefficients in a certain binomial expansion. For any series $\sum u_n$, the nth Cesàro mean of order k is defined as

$$\sigma_n^{(k)} = \frac{\sum_{\lambda=0}^{\lambda=n} A_{n-\lambda}^{(k-1)} s_\lambda}{A_n^{(k)}} = \frac{S_n^{(k)}}{A_n^{(k)}}, \tag{4}$$

and is readily seen to be a weighted mean of the partial sums $s_0, s_1, s_2, \cdots, s_n, \cdots$ of the corresponding series. These definitions may be used for any value of k not equal to a negative integer and include the positive, integral means as special cases. We consider them here for real values of $k \geqq 0$.

If we apply the linear functional operation defined by (4) to the partial sums of the series (1) for values of k in the interval $(0<k<1)$, we can no longer assert that the resulting expression is everywhere non-negative,* as in the case $k=1$. However, if we represent this expression by $\sigma_n^{(k)}(\theta)$, we find that it has the important property, in common with the expression (2), that

$$\int_0^\pi \left| \sigma_n^{(k)}(\theta) \right| d\theta \tag{5}$$

remains bounded for all n. We may speak of this property as that of being *bounded in the mean* for the interval $(0, \pi)$. In view of the evenness and periodicity of the expression $\sigma_n^{(k)}(\theta)$, it follows readily that it is bounded in the mean for any finite interval.

The ordinary partial sums of the series (1) do not have the property of being bounded in the mean for the interval $(0, \pi)$, since the set of constants

$$c_n = \int_0^\pi \left| \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt, \tag{6}$$

obtained by integrating the absolute value of the partial sums of the series over the interval $(0, \pi)$, become infinite with n approximately as the logarithm of n. This property of the constants (6) was demonstrated in 1910 by Fejér,† who named the

* See Gronwall, *On the Cesàro sums of Fourier's and Laplace's series*, Annals of Mathematics, (2), vol. 32 (1931), pp. 53–59. See in particular §2.

† Fejér, *Lebesguesche Konstanten und divergente Fourierreihen*, Journal für Mathematik, vol. 138 (1910), pp. 22–53.

set of constants obtained by multiplying (6) by $2/\pi$, Lebesgue constants. The fact that the set of Lebesgue constants is not bounded is inherently connected with the fact that the Fourier development of certain continuous functions diverges.

The boundedness in the mean of the $\sigma_n^{(k)}(\theta)$ for an interval including the origin is a determining factor in the efficacy of summation (Ck), when applied to the Fourier series of an L-integrable function.* It may also be utilized to derive certain criteria for deciding if a given set of constants are the Fourier constants of an L-integrable function, as will be shown in the ensuing discussion.

2. *Criteria for Fourier Constants.* The criteria in question involve differences of integral or non-integral order formed from an infinite set of constants. We define these as follows, for any set of constants $a_0, a_1, a_2, \cdots, a_n, \cdots$:

$$\Delta^{k+1}a_n = a_n - (k+1)a_{n+1} + \frac{(k+1)k}{1\cdot 2}a_{n+2} - \cdots . \tag{7}$$

For a non-integral value of k these differences are the limit as $p \to \infty$, provided this limit exists, of the expressions

$$\Delta_p^{k+1}a_n = a_n - (k+1)a_{n+1} + \cdots + (-1)^{p-n}\frac{(k+1)k\cdots(k-p+n+2)}{1\cdot 2\cdots(p-n)}a_p, \quad (n<p), \tag{8}$$

$$\Delta_p^{k+1}a_p = a_p.$$

If for a series Σu_n, $S_n^{(k)}$ is defined as in (4), we have the identity

$$\sum_{n=0}^{n=p} u_n a_n = \sum_{n=0}^{n=p} S_n^{(k)}\Delta_p^{k+1}a_n \tag{9}$$

by means of a transformation analogous to the well known Abel transformation of a corresponding summation, where the partial sums of the u's and the first differences of the a's are used.

* See the discussion of summability in the following papers: Chapman, *On non-integral orders of summability of series and integrals*, Proceedings of the London Mathematical Society, vol. 9 (1911), pp. 369–409, in particular §20; Gronwall, *On the summability of Fourier's series*, this Bulletin, vol. 20 (1913–14), pp. 139–146.

For any series $\sum u_n$ for which $S_n^{(k)}$ remains bounded, we may, if the a_n's satisfy the conditions*

$$\lim_{n\to\infty} a_n = 0, \qquad \sum_{n=0}^{\infty} n^k \left| \Delta^{k+1} a_n \right| < \infty, \qquad (k > 0), \tag{10}$$

allow p to become infinite in the above identity, since the expression on the right hand side approaches a limit by virtue of the restrictions imposed.† We thus obtain the result

$$\lim_{p\to\infty} \sum_{n=0}^{n=p} S_n^{(k)} \Delta_p^{k+1} a_n = \sum_{n=0}^{\infty} S_n^{(k)} \Delta^{k+1} a_n = \sum_{n=0}^{\infty} u_n a_n. \tag{11}$$

We are now ready to state our criteria in the form of a theorem.

THEOREM. *If the set of constants* $a_0, a_1, a_2, \cdots, a_n, \cdots$ *satisfy the conditions* (10), *then the series*

$$\frac{1}{2} a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots + a_n \cos n\theta + \cdots \tag{12}$$

will converge in the open interval $(0 < \theta \leqq \pi)$, *and will represent there an L-integrable function whose Fourier cosine development is given by* (12).

This theorem follows readily from the preceding discussion and the fact that for the trigonometric series (1) the expression $\sigma_n^{(k)}(\theta)$ is bounded in the mean in the interval $(0, \pi)$. Because of the conditions on the a's and the boundedness of $S_n^{(k)}(\theta)$ for the series (1) in the interval $0 < \delta \leqq \theta \leqq \pi$, we may infer the convergence of the series (12) in that interval to a function

$$f(\theta) = \frac{1}{2} a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots = \sum_{n=0}^{\infty} S_n^{(k)}(\theta) \Delta^{k+1} a_n. \tag{13}$$

The series on the right hand side of (13) converges absolutely and uniformly in the interval $0 < \delta \leqq \theta \leqq \pi$ and can therefore be integrated term by term in that interval. Thus we have

$$\int_\delta^\pi \left| f(\theta) \right| d\theta \leqq \sum_{n=0}^{\infty} \left(\int_\delta^\pi \left| \frac{S_n^{(k)}(\theta)}{A_n^{(k)}} \right| d\theta \right) \frac{A_n^{(k)}}{n^k} n^k \left| \Delta^{k+1} a_n \right|. \tag{14}$$

* The existence of the differences in the second condition follows from the first condition.

† See Chapman, loc. cit., §15.

As δ approaches zero, the right hand side of (14) approaches a definite limit in view of conditions (10) and the boundedness in the mean of $\sigma_n^{(k)}(\theta)$ in the interval $(0, \pi)$. Thus the left hand side does also, and the existence of the resulting integral shows that the function $f(\theta)$ is an L-integrable function whose Fourier cosine development is given by (12).

3. *Relationship to Criteria Previously Found.* The second condition in (10) involves a parameter k, and thus we have different criteria for different positive values of k. If we set $k=1$ in (10), the resulting conditions constitute a criterion due to Kolmogoroff.* It can be shown that the scope of our theorem increases as k diminishes. Thus any of our criteria for a positive value of $k<1$ includes Kolmogoroff's theorem. Furthermore, there is no best criterion of this type, and we have a sliding scale of tests analogous to the logarithmic scale in testing the convergence of a series of positive terms.

For the proof of these facts we need to show that if the differences $\Delta^{k+1}a_n$ satisfy the second condition of (10) for a positive value r of k, they will satisfy it for any value of k in the interval $0<k<r$. If we set $\rho=r-k$, it is readily seen that we have the formal identity

$$\Delta^{k+1}a_n = \Delta^{-\rho}(\Delta^{r+1}a_n). \tag{15}$$

The validity of the infinite process involved in this identity may be established by expressing the right hand side in the form of a double series as follows:

$$\begin{array}{llll} a_n - (r+1)a_{n+1} + \dfrac{(r+1)r}{1\cdot 2}a_{n+2} - \cdots \\ +0 \quad +\rho a_{n+1} \qquad -\rho(r+1)a_{n+2} \quad +\cdots \\ +0 \quad +0 \qquad\qquad +\dfrac{\rho(\rho+1)}{1\cdot 2}a_{n+2} - \cdots \\ +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{array} \tag{16}$$

If we form the sum, s_{pq}, of the pq terms taken from the first p rows and the first q columns of this series, the convergence of

* Kolmogoroff, *Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue*, Bulletin de l'Académie Polonaise, (A), 1923, pp. 83–86.

s_{pq} as p and q become infinite in such a manner that $q \leqq p$ follows from the existence of the left hand side of (15). For the case $q > p$, the additional terms included are seen to be negligible from the boundedness of the a's and the order of magnitude of the binomial coefficients involved. Thus (16) converges as a double series. Furthermore, the individual columns converge since their terms are all zero from a certain point on, and the individual rows converge as a result of the existence of the differences $\Delta^{r+1}a_n$. From a theorem due to Pringsheim* it now follows that the double series converges when summed by columns or rows, and that the sum by columns is equal to the sum by rows, which establishes the validity of (15).

We are now ready to show that the convergence of $\sum n^r |\Delta^{r+1}a_n|$ implies the convergence of $\sum n^k |\Delta^{k+1}a_n|$, whenever† $0 \leqq k < r$. From (15), taking account of the fact that the right-hand side is the sum by rows of (16), we have the inequality

$$|\Delta^{k+1}a_n| < |\Delta^{r+1}a_n| + \sum_{p=1}^{\infty} K p^{\rho-1} |\Delta^{r+1}a_{n+p}|, \tag{17}$$

(K a positive constant).

Hence, if we set

$$|\Delta^{r+1}a_n| = \epsilon_n / n^r, \tag{18}$$

we obtain readily the further inequality

$$\sum_{n=0}^{\infty} n^k |\Delta^{k+1}a_n| < \sum_{n=0}^{\infty} n^k |\Delta^{r+1}a_n| + \sum_{n=0}^{\infty} n^k \left(\sum_{p=1}^{\infty} \frac{K p^{\rho-1}}{(n+p)^r} \epsilon_{n+p} \right). \tag{19}$$

The single series in the first term of the right hand side of (19) is obviously convergent, k being less than r. Since, in view of our hypothesis, the ϵ_n defined by (18) is the general term of a convergent series, it will follow that the double series in the second term on the right hand side of (19) is convergent if, when we sum the expression

$$n^k p^{\rho-1} / (n+p)^r \tag{20}$$

* Bromwich, *Theory of Infinite Series*, 1st or 2d edition, §30.

† For our immediate purpose we do not need to include the case $k=0$, but the more general result follows from the same argument.

over all values of n and p for which $n+p=m$, the result is bounded for all m. This summation can be written in the form

$$(21) \qquad \frac{1}{m^r}\sum_{p=1}^{p=m} p^{\rho-1}(m-p)^k = \sum_{p=1}^{p=m}\left(\frac{p}{m}\right)^{\rho-1}\left(1-\frac{p}{m}\right)^k\frac{1}{m}.$$

For large values of m the summation on the right hand side of (21) approximates to the value of the integral

$$\int_0^1 x^{\rho-1}(1-x)^k dx,$$

and is therefore bounded for all m, as we wished to show.

We have proved that the convergence of $\sum n^r|\Delta^{r+1}a_n|$ implies the convergence of $\sum n^k|\Delta^{k+1}a_n|$, whenever $r>k\geqq 0$. That the converse is not true may be readily seen by considering the a's defined by

$$(22) \quad a_0=0,\ \lim_{n\to\infty} a_n=0; \quad a_{2n-1}-a_{2n}=0, \quad a_{2n}-a_{2n+1}=\frac{1}{n^2},$$

$(n=1, 2, \cdots)$. This set of a's satisfies the second condition in (10) for any k such that $0\leqq k<1$, but not for $k=1$.

The question naturally arises as to why we can not take the k in (10) equal to zero and thus get a more inclusive theorem than that obtained for values of $k>0$. It is readily seen, however, that if we allow the k to take on the value zero, the argument of our theorem is no longer valid, since the ordinary partial sums of the series (1) are not bounded in the mean for the interval $(0, \pi)$ but become logarithmically infinite in the mean with n, as pointed out before. A modification of (10) to meet this difficulty gives us a criterion due to Szidon,*

$$(23) \qquad \lim_{n\to\infty} a_n=0, \qquad \sum_{n=1}^{\infty}\log n\,|\Delta a_n|<\infty.$$

This latter criterion is obviously closely related to the criteria of the present paper, but it does not include them nor is it included by them.

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* Szidon, *Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen*, Mathematische Zeitschrift, vol. 10 (1921), pp. 121–127.

AN INTEGRAL EQUATION WITH SYMMETRIC KERNELS*

BY T. S. PETERSON

It is the purpose of this note to investigate conditions necessary and sufficient for the solution of the integral equation

$$(1)\quad \int_a^b A(x, s)X(s, y)ds + \int_a^b X(x, s)B(s, y)ds = C(x, y),$$

where the kernels $A(x, y)$ and $B(x, y)$ are considered to be symmetric. We further restrict our functions of two variables to be continuous throughout the fundamental interval (a, b).

An equation of the type (1) will not in general admit a solution. However, under certain quite restrictive conditions on the function $C(x, y)$, a solution may be obtained. To determine these conditions, we may readily verify from the classical theory of integral equations that every function $C(x, y)$, for which a function $X(x, y)$ exists such that (1) is true, is developable in a uniformly convergent series

$$(2)\qquad C(x, y) = \sum_{i=1}^{\infty} \left\{ \frac{\alpha_i(x)\bar{\alpha}_i(y)}{\alpha_i} + \frac{\bar{\beta}_i(x)\beta_i(y)}{\beta_i} \right\},$$

where

$$(3)\quad \bar{\alpha}_i(y) = \int_a^b \alpha_i(s)X(s, y)ds, \quad \bar{\beta}_i(x) = \int_a^b X(x, s)\beta_i(s)ds,$$

and where $\{\alpha_i, \alpha_i(s)\}$ and $\{\beta_i, \beta_i(s)\}$ are the characteristic values and characteristic functions of the kernels $A(x, y)$ and $B(x, y)$, respectively. To justify this conclusion, it is sufficient to note that the series for the iterated kernel

$$A^{(2)}(x, y) = \sum_{i=1}^{\infty} \frac{\alpha_i(x)\alpha_i(y)}{\alpha_i^2}, \qquad A^{(2)}(x, x) = \sum_{i=1}^{\infty} \frac{\alpha_i^2(x)}{\alpha_i^2},$$

converge uniformly and absolutely, which, in view of the boundedness of $\sum_{i=1}^{\infty} \bar{\alpha}_i^2(y)$, implies the uniform and absolute

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convergence of $\sum_{i=1}^{\infty}\alpha_i(x)\bar{\alpha}_i(y)/\alpha_i$. A similar argument applies for the other terms of (2).

This observation shows us that a *necessary* condition for a solution of the equation (1) under our hypotheses is that $C(x, y)$ be expressible linearly in terms of the characteristic functions of the kernels $A(x, y)$ and $B(x, y)$.

In the following theorems we shall have occasion to refer to a characteristic root of (1). *If α_i and β_i are the respective characteristic values of the symmetric kernels $A(x, y)$ and $B(x, y)$ and if $\alpha_i = -\beta_k$ (any i and any k), we say that α_i (or $-\beta_k$) is a characteristic root with respect to these kernels.*

We may then state the following theorem.

THEOREM 1. *Assuming that* (1) *has no characteristic roots and that $C(x, y)$ has the necessary form*

$$C(x, y) = \sum_{i=1}^{\infty}\{\alpha_i(x)A_i(y) + B_i(x)\beta_i(y)\}, \tag{4}$$

then if the series

$$\sum_{i=1}^{\infty}\{\alpha_i\alpha_i(x)\overline{A}_i(y) + \beta_i\overline{B}_i(x)\beta_i(y)\} \tag{5}$$

is uniformly convergent, where $\overline{A}_i(y)$ and $\overline{B}_i(x)$ are defined by

$$\begin{cases} A_i(y) = \overline{A}_i(y) + \alpha_i\int_a^b \overline{A}_i(s)B(s, y)ds, \\ B_i(x) = \overline{B}_i(x) + \beta_i\int_a^b A(x, s)\overline{B}_i(s)ds, \end{cases} \tag{6}$$

it is a solution of (1).

This theorem is established directly by the substitution of (5) into (1) and noting by our hypothesis on the characteristic roots that the functions $\overline{A}_i(y)$ and $\overline{B}_i(x)$ are determined uniquely by (6).

It is evident that a more general solution of (1) may be obtained by adding to (5) any non-vanishing solutions of the equation

$$\int_a^b A(x, s)Z(s, y)ds + \int_a^b Z(x, s)B(s, y)ds = 0. \tag{7}$$

A treatment of the non-vanishing solutions of equations essentially of the form (7) has been made by Lauricella.* We wish here only to point out the following result.

THEOREM 2. *If* (7) *has no characteristic roots, then all non-vanishing solutions of* (7) *have the property*

$$(8)\qquad \begin{cases} \displaystyle\int_a^b \alpha_i(s)Z(s, y)ds = 0, & (i = 1, 2, \cdots, n), \\ \displaystyle\int_a^b Z(x, s)\beta_i(s)ds = 0, & (i = 1, 2, \cdots, n). \end{cases}$$

We may verify this theorem at once on multiplying (7) by $\alpha_i(x)$, integrating with respect to x, and applying

$$(9)\qquad \alpha_i(x) = \alpha_i \int_a^b A(x, s)\alpha_i(s)ds.$$

This gives us

$$\left\{\int_a^b \alpha_i(s)Z(s, y)ds\right\} + \alpha_i \int_a^b \left\{\int_a^b \alpha_i(s)Z(s, t)ds\right\} B(t, y)dt = 0;$$

and hence

$$\int_a^b \alpha_i(s)Z(s, y)ds = 0.$$

A similar procedure establishes the second equation of (8).

Let us now consider the case in which there exist characteristic roots of (1). On multiplying (1) by $\alpha_i(x)\beta_k(y)$, (where $\alpha_i = -\beta_k$), and integrating on x and y, we obtain further *necessary* conditions on the coefficients of $C(x, y)$, namely,

$$(10)\qquad \int_a^b A_i(s)\beta_k(s)ds + \int_a^b B_k(s)\alpha_i(s)ds = 0.$$

With respect to these conditions, let us assume for the moment that the coefficients $B_k(s)$ are assigned arbitrarily and that the

* See Lauricella, *Sopra le funzioni permutabili di 2ª specie*, Lincei Rendconti, 1° Sem., (1913).

$A_i(s)$ are determined by (10). This means that the $A_i(s)$ will have the form

$$A_i(s) = \sum_k a_{ik}\beta_k(s) + A_i'(s), \qquad a_{ik} = -\int_a^b \alpha_i(t)B_k(t)dt, \tag{11}$$

where the summation is taken over all k for which $\alpha_i = -\beta_k$, and where $A_i'(s)$ is an arbitrary function which is orthogonal to all the corresponding $\beta_k(s)$. With this change we may rewrite the form of $C(x, y)$ as

$$\begin{aligned} C(x, y) &= \sum_{i=1}^{\infty}\{\alpha_i(x)[\sum_k a_{ik}\beta_k(y) + A_i'(y)] + B_i(x)\beta_i(y)\} \\ &= \sum_{i=1}^{\infty}\{\alpha_i(x)A_i'(y) + B_i'(x)\beta_i(y)\}. \end{aligned} \tag{12}$$

Now considering the conditions (10) for the coefficients of this new form of $C(x, y)$, we have at once, since $\int_a^b A_i'(s)\beta_k(s)ds = 0$, the additional relation

$$\int_a^b B_k'(s)\alpha_i(s)ds = 0.$$

For this reason we may assume, without loss of generality, that *a necessary condition for the existence of a solution to equation* (1) *is that the function* $C(x, y)$ *have the form* (4), *where*

$$\int_a^b A_i(s)\beta_k(s)ds = 0, \qquad \int_a^b \alpha_i(s)B_k(s)ds = 0 \tag{13}$$

for all i *and* k *such that* $\alpha_i = -\beta_k$. We may then generalize Theorem 1 to read as follows.

THEOREM 1′. *Let the function* $C(x, y)$ *have the necessary form* (4) *satisfying the additional conditions* (13); *then if the series* (5) *is uniformly convergent, it is a solution of* (1).

The only point in question with respect to the refinements made in the above theorem is whether the equations (6) will always permit a solution. We observe, however, that for $\alpha_i = -\beta_k$ the necessary and sufficient conditions for the solution of (6) are precisely the conditions (13).

Considering the equation (7) with the introduction of characteristic roots, we have a somewhat different situation. Consider, for example, that $\alpha_i = -\beta_k$. Multiplying (7) by α_i, we have

$$\alpha_i \int_a^b A(x, s)Z(s, y)ds = \beta_k \int_a^b Z(x, s)B(s, y)ds. \tag{14}$$

Multiplying (14) by $\alpha_i(x)$ and $\beta_k(y)$ separately and integrating, we have by (9)

$$\begin{cases} \displaystyle\int_a^b \alpha_i(s)Z(s, y)ds = \beta_k \int_a^b \int_a^b \alpha_i(s)Z(s, t)B(t, y)dsdt, \\ \displaystyle\int_a^b Z(x, t)\beta_k(t)dt = \alpha_i \int_a^b \int_a^b A(x, s)Z(s, t)\beta_k(t)dsdt. \end{cases} \tag{15}$$

These equations imply that

$$\int_a^b \alpha_i(s)Z(s, y)ds = \sum_k c_k\beta_k(y),$$

$$\int_a^b Z(x, t)\beta_k(t)dt = \sum_i c_i'\alpha_i(x),$$

where the summations extend over all k and i respectively for which $\alpha_i = -\beta_k$. These conditions on $Z(x, y)$ lead us to the following generalization of Theorem 2.

THEOREM 2′. *All non-vanishing solutions of* (7) *are of the form*

$$\sum_{i,k} c_{ik}\alpha_i(x)\beta_k(y) + Z(x, y), \qquad (c_{ik} = \text{const.}), \tag{16}$$

where i and k range over all indices having the property $\alpha_i = -\beta_k$, and where $Z(x, y)$ satisfies (7).

Clearly the solution, (5), of (1) is made more general by the addition of terms (16) and the sufficiency as a solution is obvious.

INSTITUTE FOR ADVANCED STUDY

ON A COVARIANT DIFFERENTIATION PROCESS: PAPER II*

BY H. V. CRAIG

1. *Introduction.* It is the purpose of this note to construct analogs of the parameters gradient, divergence, and curl, and to establish a few of their more salient properties.

2. *Notation.* In addition to the notation used in I,† we shall employ the symbols $|$, θ to indicate ordinary covariant and Synge-Taylor differentiation, respectively.

3. *The Invariants.* Evidently, if $S(x, x', \cdots, x^{(m)})$ is a scalar, then the quantities $S_{,\alpha}$ are the components of a vector. Likewise, if $V^\alpha(x, x', \cdots, x^{(m)})$, V_α $(V_\alpha = f_{\alpha\beta}V^\beta)$ are the contravariant and covariant descriptions of a vector, and $A_{\alpha\beta}$ a second order tensor, then $V^\alpha{}_{,\alpha}$ is a scalar, $V_{\alpha,\beta} - V_{\beta,\alpha}$ a skew symmetric tensor, and $A_{\alpha\beta,\gamma} + A_{\gamma\alpha,\beta} + A_{\beta\gamma,\alpha}$ a tensor of the third order. Furthermore, if n, the dimensionality of the space, is three, and $\epsilon^{\alpha\beta\gamma}$ represents the product of $|f_{\alpha\beta}|^{-1/2}$ and the corresponding component of the contravariant e system, then $\epsilon^{\alpha\beta\gamma} V_{\beta,\gamma}$ is a vector. The symbols $e^{\alpha\beta\gamma}$ are skew symmetric in each pair of indices and e^{123} is unity.

A certain regularity appears if $m > 2$ or if the affine connection is that of Riemannian geometry, for example, $V_{\alpha,\beta} = f_{\alpha\gamma}V^\gamma{}_{,\beta}$, and whenever either of these cases prevails we shall employ a special symbolism. Specifically, GS shall represent the vector $S_{,\alpha}$, DV the scalar $V^\alpha{}_{,\alpha}$, and if n is three, CV the vector $\epsilon^{\alpha\beta\gamma} V_{\beta,\gamma}$.

In virtue of these definitions and the formal equivalence of certain of the rules of operation of our process and those of partial differentiation, we may take over many of the identities of vector analysis; for example,

$$CGS \equiv 0; \qquad DCV \equiv 0; \qquad G(S + s) \equiv GS + Gs;$$

$$D(SV) \equiv (GS) \cdot V + SDV; \; C(SV) \equiv (GS) \times V + SCV; \text{ etc.}$$

The first of these relationships suggests the following theorem.

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† The preceding note, this Bulletin, vol. 37 (1931), p. 731.

THEOREM. *A necessary and sufficient condition that a function s exist such that* $S_{,\alpha}=V_\alpha$, *is* $V_{\alpha,\beta}-V_{\beta,\alpha}\equiv 0$.

In proving the sufficiency of this condition (the necessity is obvious) we shall simplify the writing by restricting ourselves to the case $m=2$. The cases $m=3$, $m>3$ are somewhat simpler and may be treated as in the following demonstration.

PROOF. Let us replace the given equations with the equivalent set of n homogeneous partial differential equations,

$$A_\alpha\Phi = 0, \quad \left(A_\alpha \equiv \frac{\partial}{\partial x'^\alpha} - 2\begin{Bmatrix}\sigma\\ \alpha\end{Bmatrix}\frac{\partial}{\partial x''^\sigma} + V_\alpha\frac{\partial}{\partial s}\right),$$

in the dependent variable Φ and the $2n+1$ independent variables x', x'', s. Obviously, these equations are independent. If, in addition, they are Jacobian complete, that is, if the alternants

$$(A_\alpha A_\beta)\Phi \equiv -\left(A_\alpha 2\begin{Bmatrix}\Lambda\\ \beta\end{Bmatrix}\right)\frac{\partial\Phi}{\partial x''^\Lambda} + V_{\beta,\alpha}\frac{\partial\Phi}{\partial s} + \left(A_\beta 2\begin{Bmatrix}\Lambda\\ \alpha\end{Bmatrix}\right)\frac{\partial\Phi}{\partial x''^\Lambda} - V_{\alpha,\beta}\frac{\partial\Phi}{\partial s}$$

vanish, then we may conclude that the set $A_\alpha\Phi=0$ has a solution,* such that $\Phi=0$ may be solved for s. The function s so obtained is the required scalar.

As a matter of fact, a demonstration that the sum $-A_\alpha 2\{{}^{\Lambda}_{\beta}\}+A_\beta 2\{{}^{\Lambda}_{\alpha}\}$ is zero is a part of the proof of the commutative property of the differentiation process,† and so $(A_\alpha A_\beta)\,\Phi$ reduces to $(V_{\beta,\alpha}-V_{\alpha,\beta})\,\partial\Phi/\partial s$, which vanishes by hypothesis.

A second theorem,‡ which indicates a similarity between divergence and $PA_{\alpha\beta\gamma}$, $(PA_{\alpha\beta\gamma}\equiv A_{\alpha\beta,\gamma}+A_{\gamma\alpha,\beta}+A_{\beta\gamma,\alpha})$, and between curl and $V_{\alpha,\beta}-V_{\beta,\alpha}$ may be stated as follows.

THEOREM. *If* $A_{\alpha\beta}$ *is a skew-symmetric tensor, then the necessary and sufficient condition that a covariant vector* Φ_α *exist, such that*

* See Goursat, *Mathematical Analysis*, vol. 2, translated by Hedrick and Dunkel, part 2, pp. 265–270; A. Cohen, *The Lie Theory of One Parameter Groups*, pp. 109–111.

† See I, p. 733.

‡ See L. P. Eisenhart, *Condition that a tensor be the curl of a vector*, this Bulletin, vol. 28 (1922), p. 425.

$$A_{\alpha\beta} = \Phi_{\alpha,\beta} - \Phi_{\beta,\alpha}, \tag{1}$$

is that

$$PA_{\alpha\beta\gamma} = 0. \tag{2}$$

PROOF. Let Φ_1 and Φ_2 be two functions which satisfy the equation $A_{12} = \Phi_{1,2} - \Phi_{2,1}$. In virtue of (2) there is a function $\Psi(x, x', x'', s)$, such that the equality

$$\frac{\partial \Psi}{\partial x'^{\alpha}} - 2\left\{ \begin{matrix} \sigma \\ \alpha \end{matrix} \right\} \frac{\partial \Psi}{\partial x''^{\sigma}} + (\Phi_{\alpha,3} - A_{\alpha 3}) \frac{\partial \Psi}{\partial s} = 0, \quad (\alpha = 1, 2),$$

holds, for the alternant in question reduces to PA_{123}. Moreover, $\Psi(x, x', x'', s) = 0$, $s = \Phi_3$ define a function Φ_3, which satisfies the required relations $A_{13} = \Phi_{1,3} - \Phi_{3,1}$; $A_{23} = \Phi_{2,3} - \Phi_{3,2}$. Evidently Φ_4, and other successive Φ functions, may be found similarly.

Finally, we note that if we transform Φ_α as a covariant vector, then both (1) and (2) will be tensor equations and consequently valid in all coordinate systems.

An additional characteristic of the invariant GS, somewhat analogous to a certain property of the gradient, namely, that the critical points of $z(x, y)$ are determined by grad $z = 0$, is expressed by the following theorem.

THEOREM. *The equation* $Gf^{(m)}(x, x') = 0$ *determines the extremal curves associated with* $\int f dt$.

PROOF. By differentiating $f(x, x')$ repeatedly with respect to the parameter and representing each time with R those terms which do not contribute to the corresponding $Gf^{(m)}$, we have

$$\begin{aligned} f' &= x'^{\alpha} f_{x^{\alpha}} + x''^{\alpha} f_{\alpha}; \\ f'' &= x''^{\alpha}(f_{x^{\alpha}} + 2x'^{\beta} f_{\alpha x^{\beta}} + x''^{\beta} f_{\alpha\beta}) + x'''^{\alpha} f_{\alpha} + R; \\ f^{(m)} &= x^{(m)\alpha}\left[f_{x^{\alpha}} + m(f_{\alpha})'\right] + x^{(m+1)\alpha} f_{\alpha} + R, \qquad (m > 2). \end{aligned}$$

Applying our process to the first of these and expanding, we obtain

$$f'_{,\gamma} = f_{x^{\gamma}} + x'^{\alpha} f_{x^{\alpha}x^{\gamma}} + x''^{\alpha} f_{\alpha\gamma} - 2f_{\sigma} f^{\rho\sigma}\left\{ x'^{\delta}[\gamma\delta, \rho] + \tfrac{1}{2} x''^{\tau} f_{\gamma\rho\tau} \right\}.$$

But the followings relations hold:

$$f_{\cdot} f^{\rho\tau} = x'^{\alpha} f_{\sigma\alpha} f^{\rho\sigma}; \; x'^{\rho} f_{\gamma\rho\tau} = 0; \; x'^{\delta} x'^{\rho}\left[f_{\rho\delta x^{\gamma}} + f_{\gamma\rho x^{\delta}} - f_{\gamma\delta x^{\rho}}\right] = 2f_{x^{\gamma}};$$

and therefore $f'_{\cdot,\gamma}$ is the Euler tensor, $-f_{x^{\gamma}} + (f_{\gamma})'$. Similarly, we find that $f^{(m)}{}_{,\gamma} = m\left[-f_{x^{\gamma}} + (f_{\gamma})'\right]$.

A second application of the process reveals the fact that the covariant Euler tensor E_α is a constant with respect to our derivative, thus:

$$E_\alpha = x''^\gamma f_{\alpha\gamma} + x'^\gamma x'^\delta [\gamma\delta, \alpha],$$

$$E_{\alpha,\beta} = x''^\gamma f_{\alpha\gamma\beta} + 2x'^\gamma [\gamma\beta, \alpha] - 2f_{\alpha\sigma} f^{\sigma\rho}(x'^\gamma [\gamma\beta, \rho] + \tfrac{1}{2} x''^\gamma f_{\rho\gamma\beta}) \equiv 0,$$

while

$$E^\alpha{}_{,\beta} = (f^{\alpha\gamma} E_\gamma)_{,\beta} = f^{\alpha\gamma}{}_{,\beta} E_\gamma = f_{x'^\beta}^{\alpha\gamma} E_\gamma = -f^{\alpha\gamma} f_{\gamma\delta\beta} E^\delta.$$

Incidentally, the tensor $f^{\alpha\gamma}{}_{x'^\beta} E_\gamma$ appears in $(\theta E_\alpha)_{,\beta}$ also, for

$$\theta E_\alpha = (x''^\gamma f_{\alpha\gamma})' + 2x''^\gamma x'^\delta [\gamma\delta, \alpha] + R - \{x''^\gamma f_{\sigma\gamma} + x'^\gamma x'^\delta [\gamma\delta, \sigma]\}\{x'^\mu [\mu\alpha, \rho] + \tfrac{1}{2} x''^\tau f_{\alpha\tau\rho}\} f^{\sigma\rho};$$

and consequently

$$(\theta E_\alpha)_{,\beta} = f_{\alpha\beta}{}' + x''^\gamma f_{\alpha\gamma\beta} + 2x'^\delta [\beta\delta, \alpha] - \{x'^\mu [\alpha\mu, \beta] + \tfrac{1}{2} x''^\tau f_{\alpha\tau\beta}\} - \tfrac{1}{2} E^\rho f_{\alpha\beta\rho} - 3 f_{\alpha\sigma} f^{\rho\sigma} \{x'^\mu [\beta\mu, \rho] + \tfrac{1}{2} x''^\tau f_{\beta\tau\rho}\}$$

$$= -\tfrac{1}{2} E^\rho f_{\alpha\beta\rho} = \tfrac{1}{2} f_{x'^\beta}^{\rho\sigma} E_\sigma f_{\alpha\rho};$$

also

$$(\theta E^\alpha)_{,\beta} = [\theta(f^{\alpha\gamma} E_\gamma)]_{,\beta} = f^{\alpha\gamma}(\theta E_\gamma)_{,\beta} = -\tfrac{1}{2} E^\rho f_{\gamma\beta\rho} f^{\alpha\gamma} = \tfrac{1}{2} E_\gamma f^{\alpha\gamma}{}_{x'^\beta}.$$

Obviously whenever the components of a tensor do not involve x'', we may make our derivative applicable by first applying the θ process. To illustrate, suppose that we have given S_{x^α}, the gradient of a scalar point function, and let us confine ourselves to Riemannian geometry—the θ process in this case reduces to Levi-Civita's derivative. Thus, by differentiating and employing the relationships $\theta x'^\gamma = E^\gamma$; $E^\gamma{}_{,\beta} = 0$, we find

$$\theta S_{x^\alpha} = x'^\gamma S_{x^\alpha|\gamma}; \quad \theta^2 S_{x^\alpha} = (\theta x'^\gamma) S_{x^\alpha|\gamma} + x'^\gamma x'^\delta S_{x^\alpha|\gamma|\delta};$$

$$(\theta^2 S_{x^\alpha})_{,\beta} = 2x'^\gamma S_{x^\alpha|\gamma|\beta} = 2x'^\gamma S_{x^\gamma|\alpha|\beta}.$$

Finally, it is interesting to note that the *curl* of $\theta^2 S_{x^\alpha}$ involves the Riemann-Christoffel tensor, thus

$$(\theta^2 S_{x^\alpha})_{,\beta} - (\theta^2 S_{x^\beta})_{,\alpha} = 2x'^\gamma S_{x^\delta} R^\delta_{\gamma\alpha\beta}.$$

THE UNIVERSITY OF TEXAS

ON THE ELEMENT OF DECOMPOSITION OF A DOUBLY PERIODIC FUNCTION OF THE SECOND KIND*

BY M. A. BASOCO

1. *Introduction.* Hermite† has shown that a meromorphic function which satisfies periodicity relations of the form

$$F(z + 2\omega) = \mu F(z), \qquad F(z + 2\omega') = \mu' F(z), \tag{1}$$

where $\omega'/\omega = a + ib$, $b > 0$, and μ, μ' are independent of z, may be expressed in terms of the function

$$G(z) = \frac{\sigma(z + \lambda)}{\sigma(z)\sigma(\lambda)} e^{\rho z}, \tag{2}$$

and its derivatives, in which λ, ρ are suitably determined constants and $\sigma(u)$ is the Weierstrass sigma function. The class of functions which satisfy conditions (1) has been called by Hermite doubly periodic of the second kind. We shall exclude from the present considerations Mittag-Leffler's singular case‡ for this category of functions. He has shown that if $F(z) = f(z)e^{\rho z}$, where $f(z)$ is an elliptic function, then the suitable element of decomposition is $e^{\rho z}\zeta(z)$, where $\zeta(z)$ is the zeta function of Weierstrass.

In what follows we shall be concerned with the nature of the power series development§ of (2) which we shall henceforth write in the form

$$\Phi(u, v) = \frac{\sigma(u + v)}{\sigma(u)\sigma(v)} e^{\rho z}, \tag{3}$$

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† Hermite, *Sur quelques applications des fonctions elliptiques*, Oeuvres, vol. 3, Chaps. 1, 2, pp. 266 et seq. See also the monograph of Appell, *Sur la décomposition d'une fonction méromorphe en éléments simples*, Mémorial des Sciences Mathématiques, Fascicule 36, 1929.

‡ Comptes Rendus, vol. 90 (1880), p. 178.

§ See Krause, *Doppeltperiodischen Funktionen*, vol. 1, pp. 299–308. Halphen, *Fonctions Elliptiques*, vol. 1, Chap. 7.

where u, v are to be regarded as independent complex variables. For suitable selections of ρ, various expressions are obtained which are of importance in the applications. Thus, if $\rho = -\zeta(v)$ or $x-\zeta(v)$, the corresponding $\Phi(u, v)$ is fundamental in certain physical problems, as will be evident from even a cursory examination of the second volume of Halphen's treatise. For $\rho = -\eta/\omega$, $(\eta = \zeta(\omega))$, the function (3) is essentially the same as $\vartheta_1'\vartheta_1(x+y)/\{\vartheta_1(x)\vartheta_1(y)\}$, which is of importance in some of the applications of doubly periodic functions to number theory.

It will be sufficient to treat in detail the case $\rho = x-\zeta(v)$; the results for the other cases mentioned being obtained on setting $x=0$, $x=Z(v)$, respectively, where this last is the zeta function of Hermite.* We find that, for this generic case, the expansion (3) in a power series in u has coefficients which are polynomials in x with elliptic functions of v as coefficients. As might be expected, these polynomials belong to the class of Appell polynomials† and can be constructed by recurrence. They will be shown to satisfy a certain linear differential equation.

2. *The Generator* ϕ. The generator for the polynomials in question is the function

$$\phi = \frac{\sigma(u+v)}{\sigma(u)\sigma(v)}\, e^{-u\zeta(v)}, \tag{4}$$

which, as may be seen from the properties of the sigma function, has a power series expansion about the origin $u=0$, of the form

$$\phi = \frac{1}{u}\left\{1 + \sum_{n=1}^{\infty} \alpha_n \frac{u^n}{n!}\right\}, \tag{5}$$

$$(\alpha_1 = 0,\ \alpha_2 = -\wp(v),\ \alpha_3 = -\wp'(v),\ \text{etc.}).$$

It follows from the theory of Appell polynomials that the function

$$\psi = \phi e^{xu} = \frac{\sigma(u+v)}{\sigma(u)\sigma(v)}\, e^{(x-\zeta(v))u} \tag{6}$$

when expanded in a power series in u will have as the coeffi-

* See Appell-Lacour, *Fonctions Elliptiques*, p. 27.

† Appell, *Sur une classe de polynômes*, Annales de l'École Normale Supérieure, 1880.

cient of the general term a polynomial $A_n(x)$ which symbolically may be represented by $(x+\alpha)^n$, where α is the umbra of the sequence $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n, \cdots$. The binomial expansion for $A_n(x)$ is to be understood in the sense that the exponents of α are to be degraded into suffixes; thus, α^k is to be replaced by α_k.

It can be readily shown* that the generator ϕ satisfies the Lamé differential equation

$$\frac{\partial^2\phi}{\partial u^2} = (2\wp(u) + \wp(v))\phi, \tag{7}$$

where $\wp(u)$ is the Weierstrass elliptic function.

3. *The Function* ψ. It follows from (6) and (7) that the function ψ satisfies the partial differential equations

$$\frac{\partial\psi}{\partial x} = u\psi, \tag{8}$$

$$\frac{\partial^2\psi}{\partial u^2} - 2x\frac{\partial\psi}{\partial u} + x^2\psi = (2\wp(u) + \wp(v))\psi. \tag{9}$$

We may assume that ψ has a power series expansion of the form

$$\psi = \frac{1}{u} + \sum_{n=0}^{\infty} A_{n+1}(x, v)\frac{u^n}{(n+1)!}. \tag{10}$$

* Halphen, vol. 1, p. 235. In this connection it may not be out of place to point out a curious error which has crept into the literature. In a paper by E. Jahnke, Journal für Mathematik, vol. 112 (1893), pp. 265–286, there are given several results concerning differential equations satisfied by doubly periodic functions of the second and third kinds. The last section of the paper (Third part, p. 282) is devoted to a study of certain differential identities due to Caspary (Journal de Mathématiques, (4), vol. 6, pp. 367–404), which give rise precisely to our generator $\phi(u, v)$. Apparently, Jahnke transcribed Caspary's function (see Caspary's paper, p. 403) erroneously and makes the statement that the function $\phi(u, v)\exp(-v\zeta(u))$ satisfies our equation (7) as well as certain others derivable from it. This, however, is not the case, as may be readily seen. Jahnke's results are correct for our generator $\phi(u, v)$ but are not true for the function

$$\frac{\sigma(u+v)}{\sigma(u)\,\sigma(v)}\exp(-u\zeta(v)-v\zeta(u))$$

used by him.

On substituting this expansion in (8), we obtain, as the result of equating the coefficients of like powers of u, the relation

$$\frac{\partial A_n}{\partial x} = nA_{n-1}, \tag{11}$$

which verifies the fact that the $A_n(x, v)$ are Appell polynomials in x.

Next, if the development (10) is substituted in (9), and both sides of the equation are expressed as power series in u, we find on equating the coefficients of corresponding powers two recurrences on the A_n, one being derived from the odd powers, the other from the even. However, if we define $A_0 = 1$, these two recurrences may be combined into the following one:

$$\begin{cases} (n-3)A_n = 2(n-2)xA_{n-1} + (n-1)(\wp(v) - x^2)A_{n-2} \\ \qquad + 2(n-1)! \sum_{k=1}^{[(n-2)/2]} C_{k+1} \dfrac{A_{n-2k-2}}{(n-2k-2)!}, \end{cases} \tag{12}$$

where

$$C_k = \frac{3}{(2k+1)(k-3)} \sum_{r=2}^{k-2} C_r C_{k-r}, \qquad (k = 4, 5, 6, \cdots),$$

and, as usual, $[s/2]$ is the greatest integer contained in $s/2$. The C_k are the coefficients in the expansion of the Weierstrass $\wp$-function.

It is clear that the coefficients α_k of the polynomials A_n are integral rational expressions in $\wp(v)$, $\wp'(v)$, and the fundamental invariants g_2, g_3 of $\wp(u)$, with rational coefficients. For example, we have

$$A_0 = 1 \text{ (by definition)}, \quad A_1 = x, \quad A_2 = x^2 - \wp(v),$$
$$A_3 = x^3 - 3\wp(v)x - \wp'(v),$$
$$A_4 = x^4 - 6\wp(v)x^2 - 4\wp'(v)x - 3\wp^2(v) + (3/5)g_2,$$
$$A_5 = x^5 - 10\wp(v)x^3 - 10\wp'(v)x^2 + 5x((3/5)g_2 - 3\wp^2(v)) - 2\wp(v)\wp'(v),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If we let $x = 0$, then recurrence (12) gives a simpler means of calculating the coefficients α_k in the generator ϕ. Moreover, if $x = Z(v) = \zeta(v) - (\eta/\omega)\, v$, where $Z(v)$ is the zeta function of Hermite, we obtain the coefficients in the development of

$$\frac{\sigma(u+v)}{\sigma(u)\sigma(r)}\,e^{-(\eta/\omega)uv},$$

which in terms of the Jacobi theta functions is*

$$\frac{1}{2\omega}\vartheta_1'\frac{\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)}, \qquad x=\frac{u}{2\omega}, \qquad y=\frac{v}{2\omega}.$$

This function is of importance in the arithmetical applications.

The differential equation of which A_n is a solution may be found readily by combining relations (11) and (12). Thus we obtain:

$$(13)\quad 2\sum_{k=1}^{[(n-2)/2]} C_{k+1}\frac{\partial^{2k+2}A_n}{\partial x^{2k+2}}+(\wp(v)-x^2)\frac{\partial^2 A_n}{\partial x^2}+2(n-2)\frac{\partial A_n}{\partial x} - n(n-3)A_n=0,$$

which is of order n or $n-1$ according as n is even or odd.

4. *The Polynomials* Γ_n. A system of Appell polynomials Γ_n analogous to the ones just discussed may be obtained from the generator

$$(14)\qquad \chi=\frac{\sigma(u+v)}{\sigma(v)}\,e^{-u\zeta(v)}=1+\sum_{n=1}^{\infty}\gamma_n\frac{u^n}{n!}.$$

The polynomials $\Gamma_n=(x+\gamma)^n$ are the coefficients in the expansion

$$(15)\qquad \Theta=\chi\exp xu=1+\sum_{n=1}^{\infty}\Gamma_n\frac{u^n}{n!},$$

and may be determined successively by means of the relation

$$(16)\qquad \Gamma_n=\frac{\partial\Gamma_{n-1}}{\partial v}-(n-1)\wp(v)\Gamma_{n-2}+x\Gamma_{n-1},$$

which is a consequence of the fact that Θ satisfies

$$(17)\qquad \frac{\partial\Theta}{\partial u}-\frac{\partial\Theta}{\partial v}=(x-\wp(v))\Theta.$$

* See Tannery-Molk, *Fonctions Elliptiques*, vol. 3, p. 137.

Further, since

$$\frac{\partial \Gamma_n}{\partial x} = n\Gamma_{n-1}, \tag{18}$$

we have, by (16), the partial differential equation satisfied by these polynomials, namely,

$$\wp(v)\frac{\partial^2 \Gamma_n}{\partial x^2} - \frac{\partial^2 \Gamma_n}{\partial x \partial v} - x\frac{\partial \Gamma_n}{\partial x} + n\Gamma_n = 0. \tag{19}$$

Moreover, it follows as a consequence of the identity

$$\Theta = \psi\sigma(u) \tag{20}$$

that the polynomials Γ_n are expressible linearly in terms of the A_n. In fact,

$$\Gamma_n = A_n + n! \sum_{r=1}^{[(n-3)/2]} \frac{b_{r+1}}{(2r+3)!}\,\frac{A_{n-2r-2}}{(n-2r-2)!} + \epsilon(n)\frac{b_{n/2}}{n+1}, \qquad (n = 1, 2, 3, \cdots), \tag{21}$$

where $\epsilon(n)$ is zero or unity according as n is odd or even, $b_{r+1}/(2r+3)!$ is the coefficient of the general power in the expansion of the sigma function about the origin,* and $A_s = 0$ in case $s < 0$. If in this result we set $x = 0$, we obtain a relation between the two umbras α and γ.

5. *Conclusion.* Other systems of Appell polynomials related to our A_n, Γ_n can be obtained from them by applying the processes indicated by Appell in his memoir.† In particular, the systems generated by the functions $d^k\phi/du^k$, $d^k\chi/du^k$ may be readily found. Thus, the general Appell polynomial generated by the second of these is given by

$$\sum_{\rho=0}^{k}(-1)^\rho\binom{k}{\rho}x^\rho\left[\left(x+\frac{\partial}{\partial v}\right)\Gamma_{n+k-\rho-1} - (n+k-\rho-1)\wp(v)\Gamma_{n+k-\rho-2}\right], \tag{22}$$

* Halphen, vol. 1, p. 300. Note that $b_1 = 0$. See also Schwarz's *Formeln und Lehrsätze*, etc., p. 7. Also, Haussner, *Über die Zahlencoefficienten in den Weierstrass'schen σ-Reihen*, Göttinger Nachrichten, 1894.

† Appell, loc. cit., Annales de l'École Normale Supérieure, 1880.

which, by (16), may be written in the form

$$\sum_{\rho=0}^{k}(-1)^{\rho}\binom{\rho}{k}x^{\rho}\,\Gamma_{n+k-\rho}, \tag{23}$$

and hence is expressible in terms of the Γ_n directly.

These results would be useful in determining the systems of Appell polynomials generated by a general doubly periodic function of the second kind.

THE UNIVERSITY OF NEBRASKA

ON DOUBLE RIEMANN-STIELTJES INTEGRALS*

BY J. A. CLARKSON

1. *Introduction.* A recent study by Clarkson and Adams† of functions $f(x, y)$ of bounded variation naturally leads one to the consideration of double Stieltjes integrals. The present paper is devoted to the discussion of certain questions concerning such integrals.

Stieltjes defined the symbol

$$\int_a^b f(x)d\phi(x) \tag{1}$$

by means of the sum

$$\sum_{i=1}^{n} f(\xi_i)[\phi(x_i) - \phi(x_{i-1})],$$

$$(a = x_0 < x_1 < x_2 < \cdots < x_n = b,\ x_{i-1} \leqq \xi_i \leqq x_i).$$

If this sum approaches a finite limit when the norm of the subdivisions approaches zero, (1) is defined as this limit; otherwise (1) is not defined. He showed that for a given $\phi(x)$, a sufficient condition that (1) should exist for every continuous function

* Presented to the Society, December 27, 1933.

† J. A. Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*, Transactions of this Society, vol. 35 (1933), pp. 824–854.

$f(x)$ is that $\phi(x)$ be of bounded variation. Pollard† has shown that this condition is also necessary.

We consider two separate extensions of this notion to functions of two variables. The first is due to Fréchet,‡ who gave the following definition. Assume that $f(x, y)$ and $\phi(x, y)$ are defined over the rectangle $R(a \leqq x \leqq b,\ c \leqq y \leqq d)$; let R be divided into rectangular subdivisions, or *cells*, by the *net* of straight lines $x = x_i,\ y = y_j,\ (a = x_0 < x_1 < x_2 < \cdots < x_m = b,\ c = y_0 < y_1 < y_2 < \cdots < y_n = d)$; let $\xi_i,\ \eta_j$ be any numbers satisfying the inequalities

$$x_{i-1} \leqq \xi_i \leqq x_i, \quad y_{j-1} \leqq \eta_j \leqq y_j,$$
$$(i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, n);$$

and for all i, j let

$$\Delta_{11}\phi(x_i, y_j) = \phi(x_{i-1}, y_{j-1}) - \phi(x_{i-1}, y_j) - \phi(x_i, y_{j-1}) + \phi(x_i, y_j).$$

Then if the sum

$$S = \sum_{i,j=1}^{m,n} f(\xi_i, \eta_j)\Delta_{11}\phi(x_i, y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to ϕ is said to exist. We call this limit the *restricted integral*, and designate it by the symbol

$$\int_a^b \int_c^d f(x, y)\, d_x d_y \phi(x, y). \tag{2}$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i,j=1}^{m,n} f(\xi_{ij}, \eta_{ij})\Delta_{11}\phi(x_i, y_j),$$

where $\xi_{ij},\ \eta_{ij}$ are any numbers satisfying the inequalities

$$x_{i-1} \leqq \xi_{ij} \leqq x_i,\ y_{j-1} \leqq \eta_{ij} \leqq y_j,$$
$$(i = 1, 2, 3, \cdots, m;\ j = 1, 2, 3, \cdots, n),$$

† S. Pollard, *The Stieltjes integral and its generalizations*, Quarterly Journal of Mathematics, vol. 49 (1920), pp. 73–138.

‡ Fréchet, *Extension au cas des intégrals multiples d'une définition de l'intégrale due à Stieltjes*, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910), pp. 241–256.

we call the limit, when it exists, the *unrestricted integral*,† and designate it by the symbol

$$(3) \qquad (*)\int_a^b \int_c^d f(x, y)d_x d_y\phi(x, y).$$

Clearly the existence of (3) implies both the existence of (2) and its equality to (3); on the other hand the existence of (2) does not imply the existence of (3), as we shall presently see.

Fréchet‡ has shown that a sufficient condition for the existence of the restricted integral (2) for every continuous integrand is that the integrator function $\phi(x, y)$ be of bounded variation in the sense of Vitali§ (we write $\phi \subset V$). Of special interest in connection with bilinear functionals is the case in which the integrand is *factorable*; that is, $f(x, y) = g(x)h(y)$. In this case Fréchet‖ has shown that a sufficient condition for the existence of the restricted integral (2) for every factorable integrand with continuous factors is that $\phi(x, y)$ be of bounded variation in a certain sense of his own‖ ($\phi \subset F$). The condition of Fréchet is known to be weaker¶ than that of Vitali.

The following questions naturally present themselves:

(a) In each case is the sufficient condition of Fréchet also necessary?

(b) In each case is the condition shown by Fréchet to be suffi-

† Such integrals have recently been considered or employed by T. H. Hildebrandt and I. J. Schoenberg, *On linear functional operations . . .*, Annals of Mathematics, vol. 34 (1933), pp. 317–328; I. J. Schoenberg, *On finite-rowed systems of linear inequalities . . .*, II, Transactions of this Society, vol. 35 (1933), pp. 452–478; and C. R. Adams, *Hausdorff transformations for double sequences*, this Bulletin, vol. 39 (1933), pp. 303–312.

‡ Fréchet, loc. cit.

§ The function $\phi(x, y)$ is said to be of bounded variation in the Vitali sense if there exists a positive quantity M such that for every net on R we have $\sum_{i,j=1}^{m,n} |\Delta_{11}\phi(x_i, y_j)| < M$; and to be of bounded variation in the Fréchet sense if there exists a positive quantity M such that we have $\sum_{i,j=1}^{m,n} \epsilon_i\bar{\epsilon}_j\Delta_{11}\phi(x_i, y_j) < M$ for every net on R and every possible choice of the numbers ϵ_i and $\bar{\epsilon}_j$ equal to $+1$ or -1.

‖ Fréchet, *Sur les fonctionnelles bilinéaires*, Transactions of this Society, vol. 16 (1915), pp. 215–234.

¶ See Littlewood, *On bounded bilinear forms in an infinite number of variables*, Quarterly Journal of Mathematics, Oxford Series, vol. 1 (1930), pp. 164–174, or Clarkson and Adams, loc. cit.

cient for the existence of the restricted integral also sufficient for the existence of the unrestricted integral?

(c) In the second case is the condition of Fréchet, employed when the integrand is factorable, sufficient for the existence of the integral (either restricted or unrestricted) when the integrand is continuous but not necessarily factorable?

We shall show (i) that the answer to both questions (a) is affirmative, which implies a negative answer to (c); and (ii) that the answer to (b) is affirmative for the first case and negative for the second.

2. *Integrals of Functions not Assumed to be Factorable.*

THEOREM 1. *A necessary (as well as sufficient) condition that the restricted integral* (2) *shall exist for every continuous* $f(x, y)$ *is that* $\phi \subset V$.

PROOF. Assuming that $\phi(x, y)$ does not satisfy this condition, we shall show that there always exists a continuous function $f(x, y)$ such that (2) does not exist.

Clearly there must exist in R at least one point P having the following property: in at least one of the four quadrants about P, ϕ is not $\subset V$ in every rectangle with sides parallel to the axes and one vertex at P. Without loss of generality we may and do assume that this is the third quadrant. Hence we infer the existence of an infinite sequence of such rectangles $\{R_p\}$, each subdivided into cells by a net N_p of lines parallel to the axes, such that

(i) the lower left corner of R_{p+1} lies in the interior of the upper right cell of N_p;

$$\text{(ii)} \qquad \sum_{(N_p)} \left| \Delta_{11}\phi(x_i, y_j) \right| > p, \qquad (p=1, 2, 3, \cdots).$$

Now define the function $f(x, y)$ as follows. At the lower left vertex of each cell of N_p, $(p=1, 2, 3, \cdots)$, let $f(x, y) = +1/p$ or $-1/p$, according as $\Delta_{11}\phi(x_i, y_j)$ is $\geqq 0$ or <0 for that cell. Let each of these vertices be surrounded by a small square $S_{ij}^{(p)}$ in such a way that all the $S_{ij}^{(p)}$ lie below a horizontal line which in turn lies below all the $S_{ij}^{(p+1)}$, and to the left of a vertical line which in turn lies to the left of all the $S_{ij}^{(p+1)}$. Now let $f(x, y)$ vanish at any point which is not in the interior of some $S_{ij}^{(p)}$,

and be so defined within these squares as to be continuous over R, which is clearly possible.

Then the integral (2) does not exist for this function. For assume that it does, and let A be its value. Then there exists a $\delta>0$ such that for any net on R, the norm of whose subdivisions is $<\delta$, we have $|A-S|<1/2$, where S is formed for this net, and any choice whatever of the intermediate points ξ_i, η_j. Let such a net N be fixed. Add to N the horizontal and vertical lines through P to form the net N', and to N' add the lines, suitably extended, of the net N_k, where k is chosen sufficiently large so that the squares $S_{ib}^{(k)}$ associated with the net N_k lie in the interior of that cell of N' which has P for upper right vertex. Let N'' be the resulting net.

Let S_1 be the sum S formed for the net N', where the points ξ_i, η_j are so chosen that $f(\xi_i, \eta_j)=0$ in all terms of S associated with those cells of N' which lie in the column immediately to the left of P, and the row immediately below P; aside from this the choice may be arbitrary. Let S_2 be the sum S formed for the net N'', with the ξ_i, η_j chosen thus: in those columns of N'' which are also columns (extended) of N_k, let the ξ_i be selected as the left-hand point of each interval. In the next column to the left of these select ξ_i such that $f(\xi_i, y)=0$ for all values of y. Analogously, in those rows of N'' which are also rows (extended) of N_k, let the η_j be selected at the bottom of each row; and in the row immediately below these, select η_j such that $f(x, \eta_j)=0$ for all x. Let the remaining points be chosen as in S_1.

Then as the norms of both N' and N'' are $<\delta$, we have $|A-S_1|<1/2$, $|A-S_2|<1/2$, and hence $|S_1-S_2|<1$; but with the specified choice of nets and intermediate points, $S_2-S_1>1$, with which contradiction the proof of Theorem 1 is complete.

COROLLARY 1. *A necessary (as well as sufficient) condition that the unrestricted integral* (3) *shall exist for every continuous* $f(x, y)$ *is that* $\phi \subset V$.

PROOF. The sufficiency of the condition may be shown by a method which does not differ in any essential from Fréchet's existence proof for the restricted integral.† The necessity follows from Theorem 1. From Theorem 1 and Corollary 1 we have at once the following result.

† See Fréchet, Nouvelles Annales, loc. cit.

COROLLARY 2. *If for a given $\phi(x, y)$ the restricted integral exists for every continuous integrand, then the unrestricted integral must exist for every continuous integrand.*

3. *Integrals of Factorable Functions.*

THEOREM 2. *A necessary (as well as sufficient) condition that the restricted integral*

$$\int_a^b \int_c^d g(x)h(y)d_x d_y \phi(x, y) \tag{4}$$

shall exist for every pair of continuous functions $g(x)$ and $h(y)$ is that $\phi \subset F$.

PROOF. Assuming that ϕ is not $\subset F$, we shall show that there always exists a pair of continuous functions $g(x)$, $h(y)$ such that (4) does not exist.

As before, we see that there must exist at least one point P (x_0, y_0) in R with the following property: in at least one of the four quadrants about P, ϕ is not $\subset F$ in every rectangle with sides parallel to the axes and one vertex at P. Again we assume the quadrant to be the third. Hence there exists an infinite sequence of such rectangles $\{R_p\}$, on each of which there exists a net N_p, such that

(i) the lower left vertex of R_{p+1} lies in the interior of the upper right cell of N_p;

$$\text{(ii)} \qquad \sum_{N_p} \epsilon_i \bar{\epsilon}_j \, \Delta_{11}\phi(x_i, y_j) > p^3, \qquad (p=1, 2, 3, \cdots),$$

for some choice of the ϵ_i's and $\bar{\epsilon}_j$'s as $+1$ or -1.

Let m_p, n_p be the number of rows and columns, respectively, in the net N_p. Let $I_i^{(p)}$, $(i=1, 2, 3, \cdots, n_p-1)$, be the segments cut off on the x-axis by the (n_p-1) left-hand-most columns, extended, of N_p, numbering serially from the left; and let $I_{n_p}^{(p)}$ be the interval lying between $I_{n_p-1}^{(p)}$ and $I_1^{(p+1)}$.

We may now define $g(x)$ as follows. Interior to the interval $I_i^{(p)}$, $(i=1, 2, 3, \cdots, n_p;\ p=1, 2, 3, \cdots)$, select any two distinct points x_{ip} and x_{ip}'; let $g(x)$ at these points assume the values $+1/p$ and $-1/p$, respectively. Let $g(x)$ vanish for $x \geqq x_0$, and be so defined for remaining values of x as to be continuous in the interval $a \leqq x \leqq b$.

Let $J_j^{(p)}$, $(j=1, 2, 3, \cdots, m_p;\ p=1, 2, 3, \cdots)$, denote the an-

alogous set of segments on the y-axis, and define the function $h(y)$ in precisely similar fashion, the points analogous to x_{ip} and x_{ip}' being denoted by y_{jp} and y_{jp}'.

Now suppose that the integral (4) exists with this choice of g and h. Let $\epsilon(>0)$ be fixed. Then there exists a $\delta>0$ such that for any net N on R the norm of whose sub-divisions is $<\delta$, and any choice whatever of the intermediate points ξ_i, η_j, the inequality

$$|S-A| = \Big|\sum_{(N)} g(\xi_i)h(\eta_j)\Delta_{11}\phi(x_i, y_j) - A\Big| < \epsilon$$

is satisfied, where A is the value of (4). Let N, such a net, be fixed. Add to N the horizontal and vertical lines through P to form the net N'. Let k be any integer $>2\epsilon$ and such that the lower left vertex of R_0 lies within the cell of N' whose upper right vertex is P. Add to N' the lines of N_k, suitably extended, to form the net N''.

Let the sum S be formed for the net N''. In those rows of cells of N'' which are not also rows (extended) of N_k let the points η_j be chosen arbitrarily; and also let an arbitrary choice be made of those ξ_i which attach to all columns of N'' except those which are also columns (extended) of N_k. In view of condition (ii) we may clearly select the remaining ξ_i and η_j from among the points $x_{ik}, x_{ik}', y_{jk}, y_{jk}'$ in such a manner that B, the part contributed to S by the cells of N_k, shall exceed k. Let S_1 denote the resulting value of the sum S. Let C denote that part of S_1 which is contributed by cells of N'' lying in rows of N'' that are also rows (extended) of N_k, excepting the cells of N_k; let D denote the part contributed by cells of N'' which occur in columns of N'' that are at the same time columns (extended) of N_k, excepting the cells of N_k; let E denote the remaining part of S_1. Then we have $S_1=B+C+D+E$ and

$$|A-B-C-D-E| < \epsilon.$$

Now form the sum S_2 by using the same net N'' and the same choice of the points ξ_i, η_j, except that those of the ξ_i which were selected from the x_{ik} and x_{ik}' are to be altered as follows: in every case where formerly a point x_{ik} was chosen, to form S_2 choose instead the point x_{ik}' which occurs in the same interval $I_i^{(k)}$, and vice versa. Then we clearly have $S_2=-B+C-D+E$, and $|A+B-C+D-E| <\epsilon$.

By repeated use of this device we easily see that the following inequalities are also satisfied:

$$|A + B + C - D - E| < \epsilon,$$
$$|A - B + C + D - E| < \epsilon.$$

From these inequalities we have at once

$$|B + D| < \epsilon, \tag{5}$$
$$|B + C| < \epsilon, \tag{6}$$
$$|C + D| < \epsilon. \tag{7}$$

But $B > k > 2\epsilon$; hence from (5) and (6) we have $D < -\epsilon$, $C < -\epsilon$, whence $|C+D| > 2\epsilon$; and as this contradicts (7), the theorem is proved.

THEOREM 3. *A necessary (as well as sufficient) condition that the unrestricted integral*

$$(*)\int_a^b \int_c^d g(x)h(y)d_x d_y\phi(x, y) \tag{8}$$

shall exist for every pair of continuous functions $g(x)$, $h(y)$ is that $\phi \subset V$.

PROOF. Again we assume ϕ is not $\subset V$, and by the same argument as before show the existence of an infinite sequence of rectangles $\{R_p\}$ with nets N_p as in the proof of Theorem 2, the condition (ii) given there being now replaced by the condition

$$\sum_{(N_p)} |\Delta_{11}\phi(x_i, y_j)| > p^3, \qquad (p = 1, 2, 3, \cdots). \tag{iii}$$

Then the functions $g(x)$ and $h(y)$ defined in that theorem will serve as a pair of functions for which the unrestricted integral (8) will not exist. Since for each p there is in every cell of the net N_p a point at which $g \cdot h = 1/p^2$, and a point at which $g \cdot h = -1/p^2$, the argument is sufficiently clear without giving the details of the proof.

COROLLARY. *If for a given $\phi(x, y)$ the unrestricted integral* (3) *exists for every factorable integrand with continuous factors, it must exist for every continuous integrand.*

BROWN UNIVERSITY

SIMILAR SEQUENCES

BY E. T. BELL

1. *Similarity.* The interesting reciprocity between the Eulerian numbers E and the pseudo-Eulerian numbers E' of Pascal,* which was noticed by Sinigallia,† is not a peculiarity of the particular sequences E, E', but is a simple instance of a general property of *all* sequences of numbers or of polynomials. According to this property, defined below and called similarity, *any* sequence is similar to an infinity of other sequences; there is no special reason for singling E' out of all the sequences similar to E.

Let $S \equiv S_n$, $S' \equiv S_n'$, $(n=0, 1, \cdots)$, be any sequences such that

$$S_n = f_n(S_0', S_1', \cdots), \qquad S_n' = f_n(S_0, S_1, \cdots),$$

for all integers $n \geqq 0$, so that the nth element S_n of S is the same function f_n of elements of S' that S_n' is of elements of S. We shall say that S, S' are *similar*, and write $S \sim S'$, and hence also $S' \sim S$.

Let $S \sim S'$, and let the relation, in symbolic or umbral notation, which enables us to express the elements of either of S, S' as functions of the elements of the other, be $R(S, S')=0$. If $R(S, S')$ is bilinear in the elements of S, S', we shall say that S, S' are *bilinearly similar*. It will be shown in §2 that all the S' bilinearly similar to any given S constitute a three-parameter family of sequences.

As an example, we state the difference equation which defines *all* E' bilinearly similar to Euler's E, using for E the notation of Lucas.‡

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* E. Pascal, Rendiconti del R. Istituto Lombardo, (2), vol. 40 (1907), pp. 461–475.

† L. Sinigallia, Rendiconti di Palermo, vol. 24 (1907), pp. 222–228.

‡ E. Lucas, *Théorie des Nombres*, Chap. 14. Sinigallia's E_{2n} is Lucas' $(-1)^n E_{2n}$. It is a great convenience in using the symbolic method to fill any gaps that may occur in a given sequence with zeros, so that the index ranges over all integers $n \geqq 0$, adding a supplementary definition to give the positions of the interpolated zeros. Thus Sinigallia's E_{2n} and Pascal's E_{2n}', $(n=0, 1, \cdots)$, would be replaced by E_n, E_n', $(n=0, 1, \cdots)$, with the supplementary defini-

The recurrence for the numbers E may be written

$$(E+2)^n + E^n - 2 = 0.$$

Let a, b, d be constants such that $ad-b^2 \neq 0$, $a+b \neq 0$. Define E' by

$$E_0' = -(b+d)/(a+b),$$

$$2a(E'+1)^{n+1} + b\left[(E'+2)^{n+1} + E'^{n+1}\right] + 2(b+2^n d) = 0.$$

It may be shown (§4) that

$$aE_0E_0' + b(E_0 + E_0') + d = 0,$$

$$a(E+E')^{n+1} + b(E^{n+1} + E'^{n+1}) = 0.$$

Hence $E \sim E'$, and the similarity is bilinear. Taking $a=2$, $b=-1$, $d=0$, we get Pascal's E', and the equivalent of Sinigallia's result. When $S \sim S'$, the elements of S, S' may be numbers, as in the above example, or polynomials (§5).

If the elements of S, S' are integers, their congruence properties (§6) may be readily investigated from the symbolic form of the similarity $S \sim S'$ by an immediate application of known theorems concerning the residues of binomial coefficients. When further there are known arithmetical properties of one of S, S', as is the case for instance with E, these may be used to reduce the congruences found as just indicated.

2. *Bilinear Similarities.* Let a, b, c, d be constants such that $ad-bc \neq 0$. In order that

$$axy + bx + cy + d = 0,$$

when solved for x, y in turn, give x the same function ϕ of y that y is of x, it is necessary and sufficient that $b=c$, and we have

---

tions $E_{2n+1}=0$, $E'_{2n+1}=0$, $(n=0, 1, \cdots)$. An account of the umbral calculus, which supplements that of Lucas, has been given in my *Algebraic Arithmetic*, 1927. In particular, the formal use of infinite processes has been justified by an abstract postulational treatment, which establishes an isomorphism with finite operations on one-rowed matrices; see also Transactions of this Society, vol. 25 (1923), pp. 135–154. We recall that if α is umbral, and a, b are ordinaries different from zero,

$$(a\alpha+b)^0 = a^0\alpha^0 b^0 = \alpha_0;\ (a\alpha+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \alpha_{n-j}.$$

Henceforth n denotes an arbitrary integer ≥ 0.

$$
\begin{aligned}
&axy + b(x + y) + d = 0, \quad ad - b^2 \neq 0, \\
&-\phi(z) = (bz + d)/(az + b), \quad x = \phi(y), \quad y = \phi(x).
\end{aligned} \tag{1}
$$

Let ξ, η be the respective umbrae of the sequences ξ_n, η_n, $(n=0, 1, \cdots)$. The generators x, y of ξ, η are $x \equiv e^{\xi t}$, $y \equiv e^{\eta t}$, where t is an indeterminate (ordinary). In order that ξ, η be bilinearly similar it is necessary and sufficient that (1) hold, and that, if one of ξ, η be given, say ξ, the successive elements $\eta_0, \eta_1, \cdots$, of η be uniquely determined by (1) when x, y are replaced by $e^{\xi t}$, $e^{\eta t}$,

$$ae^{(\xi+\eta)t} + b(e^{\xi t} + e^{\eta t}) + d = 0.$$

The last is an identity in t. Hence

$$
\begin{aligned}
&a(\xi + \eta)^0 + b(\xi^0 + \eta^0) + d = 0, \\
&a(\xi + \eta)^{n+1} + b(\xi^{n+1} + \eta^{n+1}) = 0, \qquad (n \geqq 0).
\end{aligned} \tag{2}
$$

The first of these gives

$$(a\xi_0 + b)\eta_0 + (b\xi_0 + d) = 0,$$

and the coefficient of η_{n+1} in the second is $a\xi_0 + b$. Hence a necessary and sufficient condition that $\eta_0, \eta_1, \cdots$ be determinate when ξ is given is

$$\xi_0 \neq -b/a. \tag{3}$$

When (3) is satisfied, the bilinear similarity $\xi \sim \eta$ is given by (2), and if $e^{\xi t} \equiv f(t)$, the generator $e^{\eta t}$ of any η bilinearly similar to the given ξ is, by (1),

$$-e^{\eta t} \equiv [bf(t) + d]/[af(t) + b], \quad ad - b^2 \neq 0. \tag{4}$$

Examples are given in §4.

3. *General Similarity* $\xi \sim \eta$. Let

$$x \equiv x(t) \equiv e^{\xi t}, \quad y \equiv y(t) \equiv e^{\eta t} \tag{5}$$

be the generators of any sequences ξ, η. Let $R(x, y) = 0$ be the result of eliminating t from $x \equiv x(t)$, $y \equiv y(t)$, where $x(t)$, $y(t)$ are ordinary (not symbolic) functions of t. Then $R(e^{\xi t}, e^{\eta t}) = 0$ is an identity in t. Let the coefficient of $t^n (n \geqq 0)$ in the formal Mac-Laurin expansion of $R(e^{\xi t}, e^{\eta t})$ be $R_n(\xi, \eta)$. Then

$$R_n(\xi, \eta) = 0. \tag{6}$$

Necessary and sufficient conditions for $\xi\sim\eta$ are $x=\psi(y)$, $y=\psi(x)$, and the non-vanishing of the coefficients of $\eta_0, \eta_1, \cdots$ (where ξ is the given sequence) in (6) when $n=0, 1, \cdots$, successively. The problem of determining all η such that $\eta\sim\xi$, when ξ is given, is reduced to that of finding the general formal power series solution of the functional equation $\psi(\psi(z))=z$. The linear fractional solutions were determined in §2; for the rest, see §7.

4. *Examples of Bilinear Similarities.* We shall determine all η in the respective cases

$$(\xi, \eta) = (B, B'), (R, R'), (G, G'), (E, E'),$$

where B, R, G, E are the umbrae of the numbers of Bernoulli, Lucas, Genocchi, and Euler, such that $\xi\sim\eta$, and the similarity is bilinear. The notation of Lucas, (loc. cit. Chap. 14) is used for B, R, G, E, and x, y, a, b, c, d are as in §2. Hence $ad-b^2\neq 0$. The given generators are

$$e^{Bt} = t/(e^t - 1), \qquad e^{Rt} = te^t/(e^{2t} - 1),$$
$$e^{Gt} = 2t/(e^t + 1), \qquad e^{Et} = 2e^t/(e^t + 1).$$

Consider first $(\xi, \eta)=(B, B')$. Since $B_0=1$ (in Lucas' notation for the Bernoulli numbers), the condition (3) is here $a+b\neq 0$. By (4) the generator of $\eta\equiv B'$ is

$$-e^{B't} = \frac{be^{Bt}+d}{ae^{Bt}+b} = \frac{bt+d(e^t-1)}{at+b(e^t-1)},$$

and we see in this case that the generator has a convergent MacLaurin expansion under the given condition $a+b\neq 0$. Equating coefficients of like powers of t in

$$-e^{B't}[at + b(e^t - 1)] = bt + d(e^t - 1),$$

and collecting results, we get, $(n\geqq 0)$,

$$(7) \qquad (n+2)aB'^{n+1} + b[(B'+1)^{n+2} - B'^{n+2}] + d = 0,$$
$$B_0' = -(b+d)/(a+b), \quad ad-b^2\neq 0, \quad a+b\neq 0;$$
$$a(B+B')^{n+1} + b(B^{n+1} + B'^{n+1}) = 0.$$

In the same way from the given generators for R, G we find

$$(8)\quad (n+2)a(R'+1)^{n+1} + b\left[(R'+2)^{n+2} - R'^{n+2}\right] + (n+2)b + 2^{n+2}d = 0,$$

$$R_0' = -(b+2d)/(a+2b), \quad ad - b^2 \neq 0, \quad a + 2b \neq 0;$$

$$a(R+R')^{n+1} + b(R^{n+1} + R'^{n+1}) = 0;$$

$$(9)\quad b\left[(G'+1)^{n+2} + G'^{n+2}\right] + 2aG'^{n+1} + d = 0,$$

$$G_0' = -d/b, \quad G_1' = (ad - b^2)/b, \quad ad - b^2 \neq 0, \quad b \neq 0;$$

$$a(G+G')^{n+1} + b(G^{n+1} + G'^{n+1}) = 0.$$

The results for E, E' were given in §1. The numbers G, in another notation, are the tangential coefficients of Pascal,* who defines in connection with them the pseudo-tangential coefficients β', by means of the generator (in effect)

$$e^{\beta' t} \equiv \frac{e^{2t} - 1}{4e^t - e^{2t} - 1}.$$

The β'' such that $\beta' \sim \beta''$ bilinearly are determined by the recurrence

$$(a-b)(\beta''+2)^{n+1} + 4b(\beta''+1)^{n+1} - (a+b)\beta''^{n+1} + 2^{n+1}(b-d) + 4d = 0,$$

$$\beta_0' = -d/b, \quad ad - b^2 \neq 0, \quad b \neq 0.$$

5. *Similar Sequences of Polynomials.* The simplest sequence of polynomials in u associated with a given sequence of numbers α is the sequence $A(u, \alpha)$ of Appell polynomials constructed on α,

$$A_n(u, \alpha) \equiv (u + \alpha)^n, \qquad (n = 0, 1, \cdots).\dagger$$

Let $z(t) \equiv e^{\alpha t}$ be the generator of α, and $w(t)$ that of $A(u, \alpha)$. Then, obviously, $w(t) = e^{ut}\, z(t)$. To find the sequences of polynomials $A'(u, \alpha)$ similar to $A(u, \alpha)$ we take

$$x(t) \equiv w(t) \equiv e^{ut} z(t)$$

* E. Pascal, Rendiconti di Palermo, vol. 23 (1907), pp. 358–366.

† For $\alpha = B, R, G, E$, the simplifications introduced by taking $(u+\alpha)^n$ as the nth polynomial, instead of some of the slightly different definitions in the literature, are so considerable that it might be worth while to adopt the Appell polynomials in preference to the others.

in §§2, 3, and proceed as before to find the generator $y(t)$ of $A'(u, \alpha)$, and hence the recurrence for $A'(u, \alpha)$, from which the $A_n'(u, \alpha)$ can be calculated successively. Since $A(0, \alpha) = \alpha$, these results include those for similar sequences of numbers.

As an example, we take the Euler polynomials $A_n(u, E) \equiv (u+E)^n$, and find all $A_n'(u, E)$ such that the similarity $A' \sim A$, where $A \equiv A(u, E)$, $A' \equiv A'(u, E)$, is bilinear. Proceeding as just outlined, we get

$$2a(A' + u + 1)^{n+1} + b[(A' + 2)^{n+1} + A'^{n+1}] + 2b[(u + 1)^{n+1} + 2^{n+1}d] = 0,$$

$$A_0' = -(2b + d)/(a + b), \quad a + b \neq 0, \quad ad - b^2 \neq 0;$$

$$a(A + A')^{n+1} + b(A^{n+1} + A'^{n+1}) = 0.$$

For $a=2$, $b=-1$, $d=0$, $u=0$, this gives the similarity $E' \sim E$ in §1.

6. *Congruences.* Returning to (2), (3), we assume that the ξ_n, η_n $(n \geqq 0)$ are integers.*

Let p be prime. In (2) take $n+1=p$. Then

$$(a\xi_0 + b)\eta_p + (a\eta_0 + b)\xi_p \equiv 0 \pmod{p}. \tag{10}$$

Applying the known result†

$$\binom{p-j}{h} \equiv \binom{-j}{h} \pmod{p}, \qquad (0 \leqq h < p - j < p),$$

where, by definition,

$$\binom{-m}{0} = 1, \quad \binom{-m}{n} = (-m)(-m-1)\cdots(-m-n+1)/n!, \qquad (m > 0, n > 0),$$

to (2), we get

$$(a\xi_0 + b)\eta_{p-j} + (a\eta_0 + b)\xi_{p-j} + \sum_{h=1}^{p-j-1} \binom{-j}{h} \xi_{p-j-h}\eta_h \equiv 0 \pmod{p}, \tag{11}$$

$$(p > 2, \quad 0 < p - j < p).$$

---

* *Sufficient* conditions: a, b, ad integers; $\xi_{n+1}(n \geqq 0)$ integers; $a\xi_0 + b = 1$.

† Lucas, loc. cit., §228.

The remaining possibility, $n+1>p$, is covered by the following theorem.* Let the expression of $n+1$ in the scale of p be

$$n+1 = g_s p^s + g_{s-1}p^{s-1} + \cdots + g_1 p + g_0,$$
$$0 \leqq g_j < p, \quad (j = 0, \cdots, s-1), \quad g_s \neq 0, \quad g_s < p;$$

and let λ, μ be any umbrae. Then

$$(\lambda+\mu)^{n+1} \equiv \prod_{j=0}^{s}(\lambda^{p^j} + \mu^{p^j})^{g_j} \pmod{p},$$

where all the indicated binomial expansions and subsequent multiplications are to be performed as in common algebra before exponents are lowered.

7. *The Equation* $\psi^2(x)=x$. This is a special case of Babbage's equation, on which there is an extensive literature.† Let χ^{-1} be the function inverse to χ. Then, if $\psi=g$ is any solution of $\psi^2(x)=x$, Babbage's solution is $\psi=\chi^{-1}g\chi$. But this is not the general solution, as noted by Babbage himself. Ritt determines what solutions are given by Babbage's formula.

According to a result of Leau (1), p. 6, the solution in §2 is the general solution under the conditions (a) uniformity in the entire plane, (b) no singularities other than poles or isolated essential singularities anywhere in the plane; the solution can have no essential singularities.

By the second part of a theorem of Pfeiffer (loc. cit.), if $\psi(x)$ is restricted to be analytic about the origin and to vanish there, the equation has either no solution, one, or two solutions, or an infinity of solutions.

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* Proved in my paper on *Anharmonic polynomials*, Transactions of this Society, vol. 34 (1922), p. 109.

† The following papers contain further information and references bearing on the special case (as well as the general). C. Babbage, Memoirs of the Analytical Society, 1813, Note 1, pp. 47–50. L. Leau (1), Bulletin de la Société Mathématique, vol. 26 (1898), pp. 5–9. L. Leau (2), Annales de la Faculté des Sciences de Toulouse, vol. 11 (1897), pp. E1–E110. J. R. Ritt, Annals of Mathematics, (2), vol. 17 (1915), pp. 113–122. A. A. Bennett, ibid., pp. 22–60. Several of the references in this paper relate to our problem. G. A. Pfeiffer, American Journal of Mathematics, vol. 37 (1915), p. 421. Babbage's solution is reproduced by H. Laurent, *Traité d'Analyse*, vol. 6, 1890, pp. 243–244.

THE DIOPHANTINE EQUATION $x_1x_2a_1+x_2x_3a_2+\cdots+x_nx_{n+1}a_n=\delta$

BY E. J. FINAN

The object of this paper is to prove the following theorem.

THEOREM. *If δ is the greatest common divisor of $a_1, a_2, \cdots, a_n$, then the Diophantine equation*

$$x_1x_2a_1 + x_2x_3a_2 + \cdots + x_nx_{n+1}a_n = \delta \tag{1}$$

always has a solution.

We shall first prove the following lemma.

LEMMA. *If the greatest common divisor of α and β is γ, there exist integers x and y such that $x\alpha+y\beta=\gamma$ and such that x is prime to any previously given integer d.*

If $\beta=0$ the lemma is evident. Hence assume $\beta\neq 0$. Let x and y be any pair of integers for which $x\alpha+y\beta=\gamma$. Let $\alpha=\alpha_1\gamma$ and $\beta=\beta_1\gamma$. Then

$$x\alpha_1 + y\beta_1 = 1. \tag{2}$$

This gives $y=(1-x\alpha_1)/\beta_1$. Hence x must satisfy the congruence $\alpha_1x\equiv 1$, mod β_1. Since α_1 and β_1 are relative prime, all the solutions are given by $x=x_1+k\beta_1$, where x_1 is a particular solution and k is an integer. Evidently all such values of x satisfy (2). Dirichlet's theorem on the infinitude of primes in an arithmetical progression assures us of the existence of a k that will satisfy the conditions of the lemma.

We shall now prove the theorem by induction. Since we may divide (1) by δ it is quite general to assume that the a's are relatively prime and δ is unity.

When n is even the terms of (1) may be grouped in the following manner:

$$\begin{aligned} x_2(x_1a_1 + x_3a_2) + \cdots + x_{r+1}(x_ra_r + x_{r+2}a_{r+1}) \\ + x_{r+3}(x_{r+2}a_{r+2} + x_{r+4}a_{r+3}) + \cdots \\ + x_n(x_{n-1}a_{n-1} + x_{n+1}a_n) = 1, \end{aligned} \tag{3}$$

where r is odd. We shall now determine the x's that will satisfy (3). Let δ_i be the greatest common divisor of $a_1, a_2, \cdots, a_i$, $(i=1, 2, \cdots, n)$. We shall find x_{r+4} so that (a) $x=x_{r+2}a_{r+2}+x_{r+4}a_{r+3}$ is prime to δ_{r+1} except for factors in δ_{r+3}, and (b) x_{r+4} is prime to δ_{r+3}, if (c) x_{r+2} is prime to δ_{r+1}.

Let $\delta_{n+3}=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_m^{\alpha_m}$, $a_{r+3}=\delta_{r+3}fa'_{r+3}$, and $a_{r+2}=\delta_{r+3}fa'_{r+2}$, where a'_{r+2} and a'_{r+3} are relatively prime. Let $\delta_{r+1}=\delta_{r+3}d_1$, and $p=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ be the greatest common divisor of d_1 and a'_{r+2}. Then set $a'_{r+2}=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}a''_{r+2}$, $d_1=pd_2$. But since d_2 may contain some of the p's, write $d_2=p_1^{\epsilon_1}p_2^{\epsilon_2}\cdots p_k^{\epsilon_k}d_3$, where d_3 contains none of the p's as a factor and hence is prime to a'_{r+2}. Finally let $d_3=q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4$, so that d_4 is prime to δ_{r+3}.

We assert that d_4 is the x_{r+4} we want. Evidently (b) is satisfied by the definition of d_4. We shall show that (a) is satisfied. We now have $a_{r+2}=\delta_{r+3}fp_1^{\beta_1}\cdots p_k^{\beta_k}a''_{r+2}$, $x_{r+4}=d_4$,

$$\delta_{r+1}=\delta_{r+3}p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4,$$

$(s_i=e_i+\beta_i;\ i=1, 2, \cdots, k)$. Consider

$$x=x_{r+2}\delta_{r+3}fp_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}a''_{r+2}+d_4\delta_{r+3}fa'_{r+3}$$

and

$$\delta_{r+1}=\delta_{r+3}p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4.$$

Evidently δ_{r+3} is a common factor.

If $p_i\neq q_j$, $(i=1, 2, \cdots, k;\ j=1, 2, \cdots, m)$, p_i cannot divide x because p_i is prime to d_4 and to f because δ_{r+3} is the greatest common divisor of $a_1, a_2, \cdots, a_{r+3}$, and to a'_{r+3} because p_i divides a'_{r+2} while a'_{r+2} and a'_{r+3} are relatively prime. Also d_4 is prime to x because it is prime to x_{r+2} due to the fact that d_4 is contained in δ_{r+1} and by (c) δ_{r+1} is prime to x_{r+2}. Finally, d_4 is prime to f since δ_{r+3} is the greatest common divisor of $a_1, a_2, \cdots, a_{r+3}$, and d_4 is prime to a''_{r+2} by definition of p. Hence (a) is satisfied, for the only factors common to δ_{r+1} and x are the factors q_i which are all found in δ_{r+3}.

Now by the lemma we can find x_1 and x_3 so that $x_1a_1+x_3a_2=\delta_2$ and x_3 is prime to δ_2. Then by the method given above we can find x_5 so that $x_3a_3+x_5a_4$ is prime to δ_2 except for factors in δ_4.

We continue in this way until we have determined all the x's in parentheses in (3). The quantities in these parentheses are relatively prime, for if g were a common factor it would be in δ_2. Then according to the derivation of the x's given above g can

appear in the second parenthesis as a factor only if δ_4 contains g, and in the third parenthesis only if δ_6 contains g, and so on. But since $\delta_n = 1$, g must be a unit. If the quantities in parenthesis are relatively prime it is well known that we can find $x_2, x_4, \cdots, x_n$ so that (3) is satisfied.

If n is odd the procedure is just the same with the exception that the single term in the last parenthesis is automatically prime to those in the preceding parentheses which have been determined as above, and as before we have only to determine $x_2, x_4, \cdots, x_n$.

The above theorem not only proves the existence of the x's but also enables us to calculate them.

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ON CONTINUED FRACTIONS WHICH REPRESENT MEROMORPHIC FUNCTIONS*

BY H. S. WALL

1. *Introduction.* In this paper I shall give sufficient conditions in order that a continued fraction of the form

$$\frac{b_1}{1+}\,\frac{b_2 z}{1+}\,\frac{b_3 z}{1+}\cdots \tag{1}$$

with arbitrary real or complex coefficients not zero shall represent a meromorphic function of z. Van Vleck† has shown that a sufficient condition is the following:

$$\lim_{n=\infty} b_n = 0. \tag{2}$$

Stieltjes‡ proved that (2) is necessary as well as sufficient when the b_n are real and positive. Van Vleck§ proved that when the b_n are real and $b_{2n}b_{2n+1} > 0$, and the roots of the denominators, D_{2n+1}, of the $(2n+1)$th convergents of (1) have distinct limits, not zero, for $n = \infty$, then the condition (2) is necessary and sufficient in order that (1) shall represent a meromorphic function of

* Presented to the Society, April 15, 1933.

† E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476–483.

‡ Stieltjes, Oeuvres, vol. 2, pp. 560–566.

§ E. B. Van Vleck: Transactions of this Society, vol. 4 (1903), pp. 309–310.

z. He thought, and so stated (without attempting proof), that the condition imposed upon the roots of D_{2n+1} is always satisfied when (1) is meromorphic and $b_{2n}b_{2n+1}>0$. I have supplemented Van Vleck's result (§3) by constructing an example of a continued fraction (1) with real b_n such that $b_{2n}b_{2n+1}>0$ which represents a meromorphic function of z and converges everywhere except at the poles of that function, but for which we have $\limsup|b_n|>0$.

2. *Sufficient Conditions.* We shall now develop conditions that are sufficient in order that (1) be a meromorphic function of z. If we multiply (1) by z, and make the substitution $b_1=1/a_1$, $b_n=1/(a_{n-1}a_n)$, $(n=2, 3, 4, \cdots)$, that continued fraction becomes

$$\frac{z}{a_1+}\,\frac{z}{a_2+}\,\frac{z}{a_3+}\cdots. \tag{3}$$

Let it be supposed that the a_n are arbitrary real or complex numbers different from zero, and that c is a constant so chosen that $g_n\equiv c-(a_2+a_4+\cdots+a_{2n})\neq 0$ for $n=0, 1, 2, \cdots$, $(g_0=c)$.

Put $h_n=1/g_n$, $a_1'=h_0$, $a_{2n+1}'=h_n-h_{n-1}$, $a_{2n}'=a_{2n-1}/h_{n-1}^2$, $(n=1, 2, 3, \cdots)$; and denote by A_n/B_n, A_n'/B_n' the nth convergents of the continued fractions

$$c-\frac{z}{a_1+}\,\frac{z}{a_2+}\,\frac{z}{a_3+}\cdots \tag{4}$$

and

$$\frac{1}{a_1'+}\,\frac{z}{a_2'+}\,\frac{z}{a_3'+}\cdots, \tag{5}$$

respectively. The following relations hold:

$$\begin{aligned} A_{2n-1} &= h_nA_{2n}' - A_{2n+1}', \\ B_{2n-1} &= h_nB_{2n}' - B_{2n+1}', \\ A_{2n-2} &= A_{2n-1}'/h_{n-1}, \qquad (n=1, 2, 3, \cdots), \\ B_{2n-2} &= B_{2n-1}'/h_{n-1}, \qquad (a_0=h_{-1}=1). \end{aligned} \tag{6}$$

These may be verified directly for small values of n, and the induction then completed with the aid of the fundamental recursion formulas for continued fractions.

Set

$$t_1 = 1/a_1' = 1/h_0;$$

$$t_{2n+1} = 1/(a_{2n}'a_{2n+1}') = h_{n-1}/(a_{2n-1}a_{2n}h_n), \tag{7}$$

$$t_{2n} = 1/(a_{2n-1}'a_{2n}') = h_{n-1}/(a_{2n-2}a_{2n-1}h_{n-2}),$$

$(n=1, 2, 3, \cdots)$. Then (5) takes the equivalent form

$$\frac{t_1}{1} + \frac{t_2 z}{1} + \frac{t_3 z}{1} + \cdots. \tag{8}$$

The numerators and denominators, N_n', D_n' of the nth convergent of (8) are connected with the A_n', B_n' by the relations

$$N_n' = A_n'/(a_1'a_2'\cdots a_n'), \quad D_n' = B_n'/(a_1'a_2'\cdots a_n'). \tag{9}$$

We shall need the following lemma.

LEMMA.* *If* $\lim t_n = 0$, (8) *and the equivalent continued fraction* (5) *represent a meromorphic function* $f(z)$, *and converge uniformly over every bounded closed region containing no poles of* $f(z)$. *If the series* $\sum t_n$ *converges absolutely, then there exist two entire functions* $u(z)$, $v(z)$, *without common zero, such that*

$$\lim_{n=\infty} N_n'(z) = u(z), \quad \lim_{n=\infty} D_n'(z) = v(z),$$

uniformly over every bounded region.

We are now prepared to prove the following theorem.

THEOREM 1. *The continued fraction* (3)† *in which the* a_n *are real or complex numbers different from zero, and for which the series* $\sum t_n$ (*see* (7)) *converges absolutely, has the following properties.*

(a) *The sequence of even convergents represents a meromorphic function* $F(z)$, *and converges uniformly over every closed region* R *containing no poles of* $F(z)$.

(b) *Set* $k_n = 1/(a_{2n}k_{n-1})$, $p_n = (k_n - 1)/t_{2n+1}$. *Let* R' *be an ar-*

* Perron, *Die Lehre von den Kettenbrüchen*, 1st ed., pp. 345–346.

† (3) is equivalent to (1) except for the factor z.

bitrary closed bounded region containing none of the poles of $F(z)$, and containing no limit points of the sequence $\{p_n\}$, within or upon the boundary. Then the sequence of odd convergents converges uniformly over R' to $F(z)$.

(c) *At an isolated limit point p of $\{p_n\}$, not a pole of $F(z)$, the sequence of odd convergents converges to $F(p)$ if for every infinite sequence $p_{n_1}, p_{n_2}, p_{n_3}, \cdots$ converging to p, the sequence $q_{n_m} = p_{n_m} l_{2n_m} p/(p_{n_m} - p)$, $(m = 1, 2, 3, \cdots)$, does not have the limit point -1.*

(d) *If $\{p_n\}$ has more than one limit point, and if the sequence of odd convergents converges for a value of z, it must converge to $F(z)$.*

PROOF OF (a). By (9) and the last two equations (6) it will be seen that the $2n$th convergent of (3) is identical with $c-(N'_{2n+1}/D'_{2n+1})$. Hence, under the hypothesis of the theorem, it follows by the lemma that the sequence of even convergents of (3) converges uniformly over R to the meromorphic limit $F(z) \equiv c - f(z)$.

PROOF OF (b). By (9) and the first two equations (6) the $(2n-1)$th convergent of (3) is equal to

$$c - \frac{A_{2n-1}}{B_{2n-1}} = c - \frac{k_n N'_{2n} - N'_{2n+1}}{k_n D'_{2n} - D'_{2n+1}}.$$

This may be written in the form

$$c - \frac{A_{2n-1}}{B_{2n-1}} = c - \frac{p_n N'_{2n} - z N'_{2n-1}}{p_n D'_{2n} - z D'_{2n-1}}. \tag{10}$$

From (10) and the lemma it readily follows that the sequence of odd convergents of (3) converges uniformly over R' to $F(z)$.

PROOF OF (c). Equation (10) may be written in the form

$$c - \frac{A_{2n-1}}{B_{2n-1}} = c - \frac{N'_{2n-1} + q_n N'_{2n-2}}{D'_{2n-1} + q_n D'_{2n-2}}, \tag{11}$$

where $q_n = p_n l_{2n} p/(p_n - p)$ when $z = p$. Since p is an isolated limit point of the sequence $\{p_n\}$, not a pole of $F(z)$, it follows from (10) as in the proof of (b) that any infinite sequence of odd convergents corresponding to values of n for which $\{p_n\}$ has not the limit point p must converge to $F(p)$ at $z = p$. Again, if

$p_{n_1}, p_{n_2}, p_{n_3}, \cdots$ is a sub-sequence of $\{p_n\}$ converging to p, and, as supposed, the sequence q_{n_m}, $(m=1, 2, 3, \cdots)$, does not have the limit point -1, it readily follows from (11) and the lemma that the corresponding sequence (11) converges to $F(p)$ at $z=p$. We may then conclude that the sequence of odd convergents of (3) converges to $F(p)$ at $z=p$.

PROOF OF (d). Let the sequence of odd convergents converge for a certain value of z. Since, by hypothesis, $\{p_n\}$ has more than one limit point, it must have a limit point different from z, and there is an infinity of values of n for which $|z-p_n| \geqq \delta > 0$. The sequence of odd convergents corresponding to these values of n must converge to $F(z)$, and hence the sequence itself converges to $F(z)$.

The argument used in the proof of (a) is valid under the more general hypothesis $\lim t_n = 0$. Hence if $\lim t_n = 0$, whenever the continued fraction converges it must be a meromorphic function. Now when the b_n are real and $b_{2n}b_{2n+1} > 0$, or, what is the same thing, the a_n are real and $a_{2n} \neq 0$, $a_{2n+1} > 0$, Hamburger* showed that the sequence of odd convergents always converges uniformly over an arbitrary bounded closed region R exterior to the real axis; and if the limit, $F_1(z)$, is not identical with $F(z)$, then both $F(z)$ and $F_1(z)$ are meromorphic, non-rational functions of $1/z$. Consequently if $\lim t_n = 0$, we must have $F(z) \equiv F_1(z)$, for otherwise $F(z)$ would have an infinity of poles in the vicinity of the origin, which is impossible since it is a meromorphic function of z. We therefore have the following result.

THEOREM 2. *Let the a_n of (3)† be real and $a_{2n+1} > 0$, $a_{2n} \neq 0$. Then if t_n is defined as in (7), and $\lim t_n = 0$, the continued fraction represents a meromorphic function $F(z)$, with poles lying on the real axis only. The continued fraction converges uniformly over an arbitrary closed bounded region exterior to the real axis. The sequence of even convergents converges uniformly over an arbitrary closed bounded region containing none of the poles of $F(z)$ within or upon the boundary.*

* Hans Hamburger, Mathematische Annalen, vol. 82, pp. 120–164 and 168–187.

† (3) is equivalent to (1) except for the factor z in (3). The condition $a_{2n+1} > 0$, $a_{2n} \neq 0$ is equivalent to the condition $b_{2n}b_{2n+1} > 0$.

3. *Representation of a Meromorphic Function.* We proceed to prove that a continued fraction (1) with real b_n such that $b_{2n}b_{2n+1}>0$, may represent a meromorphic function of z when $b_1, b_2, b_3, \cdots$ does not converge to 0. To construct such a continued fraction, let r, s be real and positive and $r<1$, $rs>1$, $r^3s\leqq 1$, and take $b_1=1$, and

$$b_{4n} = (1/r)(1/(r^3s))^{n/2}, \qquad b_{4n+1} = r(r/s)^{n/2},$$

$$b_{4n+2} = \sigma_n(1/(s^3r))^{n/2}, \qquad b_{4n+3} = \delta_n(1/(r^3s))^{n/2},$$

where

$\sigma_n=(1-r^{n+1})s^{n+2}/[(1-s^{n+2})s^2]$, $\delta_n=(1-r^{n+2})s^{n+2}/[(1-s^{n+2})r^2t]$, $t=(rs)^{1/2}$, $(n=0, 1, 2, \cdots)$. The continued fraction (1) with the above values of the b_n is of the kind considered by Van Vleck. For $b_{2n}b_{2n+1}>0$, $(n=1, 2, 3, \cdots)$, and $b_1=1$, it will be seen that

$$\limsup |b_n| = \begin{cases} 1/r \text{ if } r^3s = 1, \\ \infty \text{ if } r^3s < 1. \end{cases}$$

If we substitute kz for z, b_n' for kb_n, $k>0$, we will obtain a continued fraction of the same kind but for which $\limsup|b_n'|$ can be made any positive number by choosing k properly.

To prove that the continued fraction represents a meromorphic function, we shall apply Theorem 1. Put $b_1=1/a_1$, $b_n=1/(a_{n-1}a_n)$, $(n=2, 3, 4, \cdots)$, so that

$$a_{2n}=(b_1b_3b_5\cdots b_{2n-1})/(b_2b_4b_6\cdots b_{2n}), \qquad (n=1, 2, 3, \cdots).$$

I find that if $c=1/(1-r)$,

$$a_{4n+2} = ct^{n^2+3n}(1-s^{n+2}), \qquad (n=0, 1, 2, \cdots),$$

$$a_{4n} = ct^{n^2+3n}(1-r^{n+1})/r^{n+1}, \qquad (n=1, 2, 3, \cdots),$$

$$h_{2n} = 1/(ct^{n^2+3n}),\ h_{2n-1} = r^{n+1}/(ct^{n^2+3n}),$$

$$k_{2n} = 1/(1-r^{n+1}),\ k_{2n-1} = 1/(1-s^{n+1}),$$

$$p_{2n} = r^{n+1}t^n/(1-r^{n+1}),\ p_{2n-1} = s^{n-1}t^{n-1}/[\sigma_{n-1}(1-s^{n+1})],$$

$$t_{4n+1} = 1/t^n = t_{n+2},\ t_{4n+3} = \sigma_n s^2/t^n,\ t_{4n+4} = \delta_n/(s^2t^{3n}),$$

$$q_{2n} = \delta_{n-1}r^2tz/\{[r(r^3s)^{n/2}+(r^{n+1}-1)z]s^{n+1}\}.$$

Since $\lim t_n=0$, Theorem 2 may be applied to prove that the continued fraction (3) represents a meromorphic function $F(z)$ and converges uniformly over every closed bounded region exterior to the real axis. This theorem gives us no information re-

garding the convergence of the sequence of odd convergents for points on the real axis. We therefore turn to Theorem 1, which is applicable here inasmuch as $\sum t_n$ converges absolutely.

When $r^2s<1$, the sequence $\{p_{2n}\}$ converges to 0, while $\{p_{2n+1}\}$ increases numerically to ∞. Hence by (a) and (b) of Theorem 1, the continued fraction (3) converges uniformly over an arbitrary bounded closed region R not containing the origin or a pole of $F(z)$. Since $q_{2n}=p_{2n}t_{4n}p/(p_{2n}-p)=0$ when $p=0$, so that $\{q_{2n}\}$ does not have the limit point -1, and since 0 is an isolated limit point of $\{p_n\}$, (c) of Theorem 1 is applicable at $z=0$.

When $r^2s=1$, $\lim p_{2n}=r$, $\lim |p_{2n+1}|=\infty$; and when $p=r$, $\lim q_{2n}=0$. Hence in this case also the continued fraction (3) converges except at the poles of $F(z)$.

Now (1) and (3) are equivalent except for the factor z in (3). It follows that if $z\neq 0$, (1) represents the meromorphic function $H(z)=F(z)/z$, and converges except at the poles (not $z=0$) of $H(z)$. It will be seen that $z=0$ is not a pole of $H(z)$, but that $H(0)=b_1=1/a_1$. Since (1) converges to this limit when $z=0$, it follows that the point $z=0$ is not an exceptional point.

When $r^2s<1$, lim sup $|b_n|=\infty$, and, by a theorem of Van Vleck,* the distance between the origin and the nearest root of D_n (the denominator of the nth convergent of (1)) has no positive lower limit. It is interesting to exhibit this fact directly for the continued fraction under consideration. By (9) and (6), it will be seen that the roots of D_{4n-1} are the roots of

$$P_{2n}D'_{4n}-zD'_{4n-1}. \tag{12}$$

Now this converges uniformly over a region containing the origin, and, since $\lim p_{2n}=0$, its limit is $-zv(z)$, a function which has a zero at the origin. We conclude that an arbitrarily small vicinity of the origin must contain a zero of (12), and hence of D_{4n-1}, provided n is sufficiently large.

When $r^2s=1$, lim sup $|b_n|=1/r$, and, by this same theorem of Van Vleck, the roots of D_n cannot come within some minimum distance $d>0$ of the origin.

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* Transactions of this Society, vol. 4 (1903), p. 209.

SOME NEW THEOREMS ON LIMITS OF VARIATION*

BY R. L. PEEK, JR.

1. *Introduction.* The application of statistical methods to engineering and inspection data makes it possible to describe the variability of such data in quantitative terms. It is often impractical to attempt a complete description of the variability, such as would be given by the frequency distribution of the quantity in question, and it suffices in many cases to state limits to the variations from the arithmetic mean such that variations exceeding these limits occur with a frequency less than some specified amount. In a practical methodology the evaluation of such virtual upper limits to the variations requires the use of expressions which are independent of the type of frequency distribution involved, as the form of the latter will often be unknown.

The best known expression for such an upper limit is that given by Tchebycheff's theorem. While the limits given by this theorem are as close as any that can be stated with perfect generality in terms of the standard deviation alone, they are much larger than those applying to most frequency distributions of common occurrence in engineering data. There is presented below a new theorem, as general in its application as Tchebycheff's, by means of which much closer limits can be placed on the relative frequency of results differing from the arithmetic mean by more than a specified amount. This more stringent inequality involves the use of both the standard deviation and the average deviation.

2. *Folded Distributions.* It will be convenient in the following discussion to employ the concept of *folding* a frequency discussion. By considering the original distribution as referred to the arithmetic mean as origin, any member of the distribution will be denoted by its deviation from the mean, x, and its relative frequency by p. The absolute values of these deviations $x_1 = |x|$, constitute a new frequency distribution for which

* Presented to the Society, June 22, 1933.

$p(x_1)=p(x)+p(-x)$. This derived distribution may be described as the *folded* distribution, for it may be obtained by folding a graphical representation of the original distribution about its arithmetic mean, and adding the superimposed ordinates. The arithmetic mean, $\bar{x}_1$, of the *folded* distribution is, by definition, the average deviation of the original distribution.

Similarly, the *folded* distribution can itself be *folded*, yielding a second derived distribution, of which the members are denoted by $x_2=|x_1-\bar{x}_1|$, and which may be described as the second folded distribution. The standard deviation of the original distribution will be written as σ, that of the first folded distribution as σ_1, and so on.

The inequalities presented below follow from a consideration of the limits of variation in these folded distributions. There is required a preliminary theorem, which may be proved by direct substitution in the expression for σ_1 of the expressions for σ and $\bar{x}_1$. This theorem states that:

If σ is the standard deviation of any set of numbers (the original distribution), $\bar{x}_1$ their average deviation, and σ_1 the standard deviation of their absolute deviations from the mean (the folded distribution), then

$$\sigma^2 - \bar{x}_1^2 = \sigma_1^2 \ . \tag{1}$$

3. *The New Inequality.* With the aid of the preceding theorem, it can be shown, by a proof similar to that for Tchebycheff's theorem, that the following is true.

If σ is the standard deviation of any set of numbers, and $\rho=\bar{x}_1/\sigma$, where $\bar{x}_1$ is the average deviation, then the relative frequency, $1-P_{t\sigma}$, of numbers differing from the mean by more than $t\sigma$ obeys (for $t>\rho$) the following inequality:

$$1 - P_{t\sigma} \leqq \frac{1-\rho^2}{t^2 - 2t\rho + 1} \ . \tag{2}$$

Considering the folded distribution, we see readily that σ_1^2 must exceed $(1-P_{t\sigma})(t\sigma-\bar{x}_1)^2$, this being the least contribution to σ_1^2 that can be made by the fraction, $1-P_{t\sigma}$, of the numbers which exceed $t\sigma$. Of the remaining fraction, $P_{t\sigma}$, of the numbers, expressed as deviations from the mean, the absolute

values of some at least must be less than $\bar{x}_1$. It will next be shown that the least contribution which the fraction $P_{t\sigma}$ of the numbers can make to σ_1 is that made in the special case in which all these numbers have the absolute value z_1, where $P_{t\sigma}(\bar{x}_1 - z_1) = (1 - P_{t\sigma})(t\sigma - \bar{x}_1)$. This leaves the value of $\bar{x}_1$ unchanged, if all the numbers included in the fraction $1 - P_{t\sigma}$ have the absolute value $t\sigma$, and it is evident that if some of the numbers included in the fraction $1 - P_{t\sigma}$ have absolute values greater than $t\sigma$, z_1 must be smaller for $\bar{x}_1$ to be unchanged, and the contribution of the fraction $P_{t\sigma}$ to σ_1 will thereby be increased. It remains to be shown that if some of the numbers included in the fraction $P_{t\sigma}$ have absolute values other than z_1, σ_1 will thereby be increased.

Suppose a group of numbers having a relative frequency q' to be changed in absolute value from z_1 to some other value z_1'. For $\bar{x}_1$ to be unchanged in value, another group of numbers having the frequency q'' must be changed to z_1'', where

$$(3) \qquad q'(\bar{x}_1 - z_1') + q''(\bar{x}_1 - z_1'') = (q' + q'')(\bar{x}_1 - z_1).$$

Now before the change, these numbers contributed to σ_1^2 an amount $(q' + q'')(\bar{x}_1 - z_1)^2$, while after the change they contribute an amount $q'(\bar{x}_1 - z_1')^2 + q''(\bar{x}_1 - z_1'')^2$, which, from (3), exceeds the former amount by $q'q''(z_1' - z_1'')^2/(q' + q'')^2$. Hence any change from the special case cited results in an increase to the contribution to σ_1^2 made by the fraction $P_{t\sigma}$ of the numbers. It follows that

$$\sigma_1^2 \geqq (1 - P_{t\sigma})(t\sigma - \bar{x}_1)^2 + P_{t\sigma}\left(\frac{(1 - P_{t\sigma})(t\sigma - \bar{x}_1)}{P_{t\sigma}}\right)^2.$$

Substituting in this the value of σ_1 given by (1), and writing ρ for $\bar{x}_1/\sigma$, gives (2) above.

4. *A Corollary of the Preceding Theorem.* The preceding theorem has been obtained by determining the minimum value of σ_1, the standard deviation of the folded distribution, composed of the absolute deviations from the mean of the original distribution. The same treatment may be applied to each of the distributions formed by taking successives folds. Thus the ap-

plication to the second folded distribution of the preceding argument gives the inequality

$$1 - P_{t\sigma} \leqq \frac{1 - \rho^2 - \rho'^2}{t^2 - 2t(\rho + \rho') + 2\rho\rho' + 1}, \tag{4}$$

where $\rho' = \bar{x}_2/\sigma$. The argument may be readily generalized to apply to the distribution of the nth fold, and if ρ_n is written for $\bar{x}_n/\sigma$, a general expression for $1 - P_{t\sigma}$ may be obtained from (4) by writing the sum of the ρ's for $\rho + \rho'$, the sum of their squares for $\rho^2 + \rho'^2$, and the sum of their paired products for $\rho\rho'$.

5. *Limits to Variation in a Continuous Distribution.* The preceding theorem is primarily an application to the quantity $\sigma^2 - \bar{x}_1^2$ of the argument leading to Tchybecheff's theorem. Similarly, an inequality applying to continuous frequency distributions may be obtained by applying to this same quantity the argument employed in developing the Camp-Meidell inequality. There may thus be obtained the following theorem.

For a frequency distribution such that the probability P of a result differing from the arithmetic mean by less than x_1 is a continuous function for which d^2P/dx_1^2 is always negative for $x_1 > \bar{x}_1$ (the average deviation), and for which σ is the standard deviation and $\rho = \bar{x}_1/\sigma$, we have

$$1 - P_{t\sigma} < \frac{4(1 - \rho^2)}{9(t - \rho)^2}. \tag{5}$$

This is most readily proved by reference to a geometrical representation of the relation between P and x_1, such as that shown in Fig. 1. As the slope of this curve is, by definition, the sum of the ordinates at x_1 and $-x_1$ of the original frequency distribution curve, σ_1^2 may be written

$$\sigma_1^2 = \int_0^1 (x_1 - \bar{x}_1)^2 dP.$$

Referring to Fig. 1, it is evident that the value of σ_1^2 will exceed the value of this last integral over the (shaded) area beneath the tangent to P at $t\sigma$. Writing the slope of this tangent in terms of the intercept P_0 on $x_1 = \bar{x}_1$, we have

$$(6) \qquad \sigma^2 - \bar{x}_1^2 > \int_{P_0}^{1} \frac{(t\sigma - \bar{x}_1)^2}{(P_{t\sigma} - P_0)^2}(P - P_0)^2 dP$$

$$= \frac{(t\sigma - \bar{x}_1)^2(1 - P_0)^3}{3(P_{t\sigma} - P_0)^2}.$$

Now for any distribution meeting the conditions stated in the theorem, the above inequality will hold if P_0 is taken as having that value for which the integral of (6) is a minimum. This value of P_0 is, of course, that for which the partial deriva-

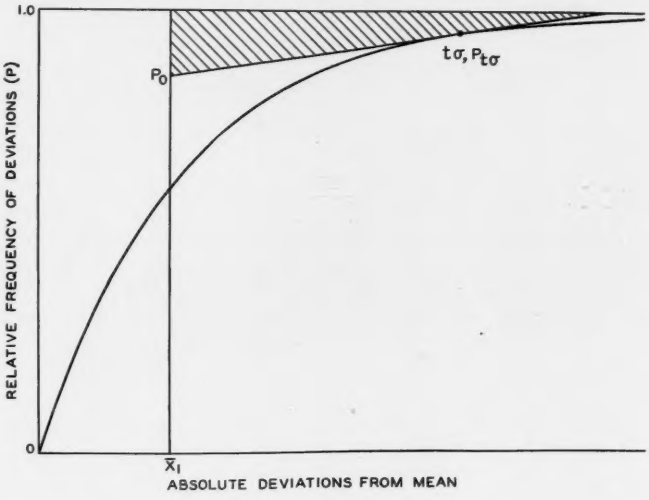

FIG. 1

tive of the right hand side of (6) with respect to P_0 is zero, and is thus found to equal $3P_{t\sigma} - 2$. Substituting this latter quantity in (6) gives the minimum value for σ_1 for any distribution meeting the conditions stated above, and thus leads directly to (5).

6. *Discussion.* The closeness of the limits given by (2) depends upon the value of ρ for the frequency distribution in question. This quantity can have any value from zero to unity. Values for several familiar types of distribution are listed in Table I.

TABLE I

Distribution	ρ	$1-\rho^2$
Two numbers of Equal Frequency	1	0
Normal Law	$(2/\pi)^{1/2}=0.798$	0.363
3 Term Gram Charlier Series	$(2/\pi)^{1/2}\left(1-\frac{\beta_2-3}{24}\right)$	$\frac{\pi-2}{\pi}+\frac{\beta_2-3}{12\pi}\left(2-\frac{\beta_2-3}{24}\right)$
Rectangular	$3/2^{1/2}=0.867$	0.250
Right Triangular	$(16/27)\cdot 2^{1/2}=0.838$	0.298
Distribution for which (in above notation) $P=kx_1^n$	$(2n+1)^{1/2}/(n+1)$	$[n/(n+1)]^2$

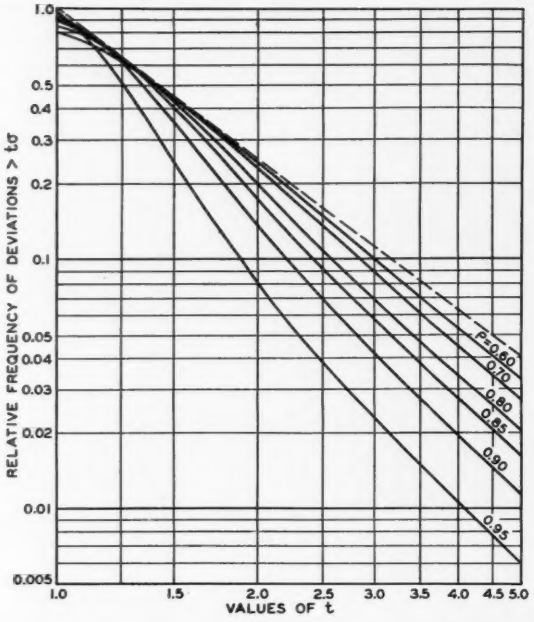

FIG. 2

Comparison of (2) with Tchebycheff's theorem shows that ρ is really a measure of the flatness of a distribution, and this is illustrated by the values of ρ listed in Table I. To afford a com-

parison of the limits given by (2) and by Tchebycheff's theorem, Fig. 2 has been prepared. This shows the values of $1-P_t$ plotted against t, the several curves corresponding to various values of ρ as indicated. The dotted line gives the limits from Tchebycheff's theorem.

BELL TELEPHONE LABORATORIES

CHARACTERISTICS OF MULTIPLE CURVES AND THEIR RESIDUALS*

BY T. R. HOLLCROFT

Salmon† obtained formulas relating the characteristics of two curves which together form the complete intersection of two algebraic surfaces when one of the curves is double on one of the surfaces. In this paper, by a generalization of Salmon's method, the relations between the characteristics of two such curves are found when one of the curves is of given multiplicity on each of the two surfaces. Such a formula is useful in studying a system of surfaces with a multiple basis curve. It was this need for it that led to its derivation.

Consider two algebraic surfaces f_1 and f_2 of orders μ_1 and μ_2, respectively, whose complete intersection consists of two curves C_1, C_2 of orders n_1, n_2; ranks r_1, r_2; genera p_1, p_2; and with h_1, h_2 apparent double points, respectively. Assume that C_1 is of multiplicity i_1 on f_1 and i_2 on f_2 and also that C_1 itself is the complete intersection curve of two surfaces. C_1 (counted simply) and C_2 have t actual intersections and n_1n_2-t apparent intersections.

Consider a third surface f_3 of order μ_3 passing simply through C_1 but not through C_2. The equivalence E of C_1 on the three surfaces f_1, f_2, f_3 is‡

$$E = n_1(i_2\mu_1 + i_1\mu_2 + i_1i_2\mu_3 - 2i_1i_2) - i_1i_2r_1.$$

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*Presented to the Society, April 14, 1933.

† Salmon, *Geometry of Three Dimensions*, 4th ed., 1882, p. 322.

‡ M. Noether, *Sulle curve multiple di superficie algebriche*, Annali di Matematica, (2), vol. 5 (1871), p. 166.

The number of points of intersection of f_1, f_2, f_3 outside of C_1 is, therefore, $\mu_1\mu_2\mu_3 - E$.

The surfaces f_1 and f_2 intersect in the curve C_1 counted i_1i_2 times and the residual curve C_2 of order $n_2 = \mu_1\mu_2 - i_1i_2n_1$. Since f_3 does not pass through C_2, the number of intersections of f_1, f_2, f_3 outside of C_1 equals the number of intersections of C_2 and f_3 less the number t of actual intersections of C_1 and C_2. Equating the two expressions for the number of intersections of f_1, f_2, f_3 outside of C_1, we have

$$\mu_3(\mu_1\mu_2 - i_1i_2n_1) - t = \mu_1\mu_2\mu_3 - E,$$

and solving for t, we find

$$\text{(A)} \qquad t = n_1(i_2\mu_1 + i_1\mu_2 - 2i_1i_2) - i_1i_2r_1.$$

In order to find an expression for t in terms of the characteristics of C_2, consider the locus F of points whose polar planes with respect to f_1 and f_2 meet an arbitrary line at one point. This surface F is of order $\mu_1+\mu_2-2$.* Since the curve C_1 is i_1-fold on f_1 and i_2-fold on f_2, it is of multiplicity i_1+i_2-2 on F.

The residual curve C_2 meets F in $n_2(\mu_1+\mu_2-2)$ points. Since F contains C_1 as an (i_1+i_2-2)-fold curve and since the surfaces f_1 and f_2 have contact at each of the t actual intersections of C_1 and C_2, these intersections count as $(i_1+i_2-1)t$ intersections of C_2 and F. Also, C_2 and F intersect in the r_2 points of contact of the r_2 tangents to C_2 which meet the arbitrary line associated with F. Equating the two expressions for the number of intersections of C_2 and F, we find

$$\text{(B)} \qquad (i_1 + i_2 - 1)t + r_2 = n_2(\mu_1 + \mu_2 - 2).$$

Eliminating t from (A) and (B) and simplifying, we obtain

$$\text{(1)} \qquad \begin{aligned} r_2 - i_1i_2(i_1 + i_2 - 1)r_1 &= n_2(\mu_1 + \mu_2 - 2) \\ &\quad - n_1(i_1 + i_2 - 1)(i_2\mu_1 + i_1\mu_2 - 2i_1i_2). \end{aligned}$$

Substituting $r_k = n_k(n_k-1) - 2h_k$, $(k=1, 2)$, in (1), we find the following relation involving apparent double points:

$$\text{(2)} \qquad \begin{aligned} 2h_2 - 2i_1i_2(i_1 + i_2 - 1)h_1 &= n_1(i_1 + i_2 - 1)[i_2\mu_1 + i_1\mu_2 \\ &\quad - i_1i_2(n_1 + 1)] - n_2(\mu_1 + \mu_2 - n_2 - 1). \end{aligned}$$

* Salmon, loc. cit., pp. 308–309.

Substituting $2h_k=(n_k-1)(n_k-2)-2p_k$, $(k=1, 2)$, in (2), we obtain the following relation involving the genus:

$$2(p_2-1)-2i_1i_2(i_1+i_2-1)(p_1-1)=n_2(\mu_1+\mu_2-4) \\ -n_1(i_1+i_2-1)(i_2\mu_1+i_1\mu_2-4i_1i_2). \tag{3}$$

Formulas (1), (2), (3) define the characteristics of either curve when those of the other and the multiplicities of C_1 are known. The order and multiplicities of C_1 must satisfy the inequality

$$n_1i_k(i_k-1)\leqq(\mu_k-1)(\mu_k-2), \qquad (k=1, 2).^*$$

When the multiple curve C_1 is the complete intersection of f_1 and f_2, the equation $\mu_1\mu_2=i_1i_2n$ and the above formulas become (on setting $n_2=r_2=0$ and dropping the subscripts of the characteristics of C_1):

$$i_1i_2r=n(i_2\mu_1+i_1\mu_2-2i_1i_2), \tag{1'}$$

$$2i_1i_2h=n(\mu_1-i_1)(\mu_2-i_2), \tag{2'}$$

$$2i_1i_2(p-1)=n(i_2\mu_1+i_1\mu_2-4i_1i_2). \tag{3'}$$

The above methods and formulas are valid for all values of order and multiplicity satisfying the necessary conditions, provided that C_1 counted i_1i_2 times and C_2 together constitute the complete intersection of f_1 and f_2 and provided further that C_1 is itself a complete intersection of two surfaces.

The assumption that C_1 is the complete intersection of two surfaces is necessary because the equivalence formula used for C_1 was derived by Noether† only for a complete intersection curve. However, it has been stated by Hudson,‡ on evidence not sufficient for proof, that this equivalence formula is very probably valid for all space curves of given order and rank whether complete intersections or not. It is, therefore, probable that the formulas derived above hold for all space curves, but no proof of this is available.

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* For $i_k>2$ and large values of μ_k, this inequality must be replaced by another. See T. R. Hollcroft, *Multiple points of algebraic curves*, this Bulletin, vol. 35 (1929), pp. 848–849.

† M. Noether, loc. cit.

‡ H. Hudson, *Cremona Transformations*, 1927, p. 221, end of §15.

ON CUBIC CONGRUENCES*

BY H. R. BRAHANA

1. *Introduction.* In the consideration of metabelian groups G of order p^{n+3} which contain a given abelian group H of order p^n and type 1, 1, $\cdots$, there enters an irreducible cubic congruence

$$x^3 + \gamma x^2 - \alpha x + \beta \equiv 0, \pmod{p}. \tag{1}$$

It is necessary to determine how many congruences (1) are distinct under certain transformations on the generators of G.†

Let the generators of H be $s_1, s_2, \cdots, s_n$, and let U_1, U_2, U_3 be three operators of order p from the group of isomorphisms of H. Let the operators $s_1, \cdots, U_3$ be permutable except for the relations

$$\begin{aligned} &U_1^{-1}s_1U_1 = s_1s_3,\ U_2^{-1}s_1U_2 = s_1s_5,\ U_3^{-1}s_1U_3 = s_1s_3^{\alpha}s_4^{\beta}s_5^{\gamma}, \\ &U_1^{-1}s_2U_1 = s_2s_4,\ U_2^{-1}s_2U_2 = s_2s_3,\ U_3^{-1}s_2U_3 = s_2s_5. \end{aligned} \tag{2}$$

Such operators U_1, U_2, U_3 obviously exist. The condition that $\{U_1, U_2, U_3\}$ contain no operator permutable with any operator, except identity, of $\{s_1, s_2\}$ is readily seen to be that (1) be irreducible, (mod p).

The group $G=\{H, U_1, U_2, U_3\}$ is a subgroup of the holomorph of H. For the sake of simplicity in the subsequent computations we shall show that generators of G may be chosen so that $\gamma=0$, provided $p>3$. Let $s_1' = s_1s_2^{-1}$, $U_2' = U_1^{-1}U_2$, and $U_3' = U_1U_2^{-2}U_3$. The operators $s_1', s_2, \cdots, s_n, U_1, U_2', U_3'$ generate G, and satisfy (2) with new numbers α', β', γ', where

$$\begin{aligned} \alpha' &= 2\gamma + \alpha - 3, \\ \beta' &= \alpha + \beta + \gamma - 1, \\ \gamma' &= \gamma - 3. \end{aligned}$$

Hence by repeating this transformation we may reduce γ to

* Presented to the Society, October 28, 1933.

† See my paper, *On the metabelian groups which contain a given group H as a maximal invariant abelian subgroup.* This paper has been offered to the American Journal of Mathematics.

zero provided $p \neq 3$. We may therefore assume that γ in (2) is zero. The congruence (1) then becomes

$$x^3 - \alpha x + \beta \equiv 0, \pmod{p}. \tag{3}$$

There are many pairs α, β such that (3) is irreducible and each such pair determines a group of the type in question. It is necessary to determine how many of these groups are distinct. We shall show that there is just one such group by proving the following theorem.

THEOREM 1. *In any given group whose generators satisfy* (2) *with* $\gamma = 0$ *and no relations which are not consequences of* (2), *generators may be chosen to satisfy* (2) *with* $\gamma' = 0$ *and* α' *and* β' *the coefficients of an arbitrary irreducible cubic of the form* (3).

We consider the following choice of generators of G, not the most general on account of the complexity of the required computations. We take

$$\begin{aligned} s_1' &= s_1 s_2^a, \qquad s_2' = s_1^b s_2^c, \\ U_1' &= U_1, \qquad U_2' = U_1^k U_2^l U_3^m, \qquad U_3' = U_1^{k_1} U_2^{l_1} U_3^{m_1}. \end{aligned} \tag{4}$$

We require $a, \cdots, m_1$ to be such that the new generators satisfy (2), with $\gamma = 0$. The transformation misses complete generality only because of U_1'. The requirement that the commutator of U_2' and s_2' be the same as that of U_1' and s_1' gives

$$\begin{aligned} bk + cl + \alpha bm &\equiv 1, \\ ck \qquad + \beta bm &\equiv a, \\ bl + \ cm &\equiv 0, \pmod{p}. \end{aligned}$$

Solving this system for k, l, and m, we have

$$\begin{aligned} k &= [a\rho + \beta b^2(c - ab)]/(c\rho), \\ l &= c(c - ab)/\rho, \\ m &= -b(c - ab)/\rho, \end{aligned} \tag{5}$$

where $\rho = c^3 - \alpha b^2 c + \beta b^3$, which cannot be zero because (3) is irreducible. The operator $U_2' = U_1^k U_2^l U_3^m$ is not a power of U_1' because s_1' and s_2' are independent, which makes $c - ab$ different from zero. We may now compute the exponents of $s_6' = s_3^r s_4^q s_6^t$, the commutator of U_2' and s_1'. They are

$$r = \{a\rho + (c - ab)[\beta b^2 + ac^2 - \alpha bc]\}/(c\rho),$$
$$s = [a^2\rho - \beta b(c - ab)^2]/(c\rho),$$
$$t = (c - ab)^2/\rho.$$

Computing the commutator of U_3' and s_2' and requiring it to be s_3', we get the system

$$\begin{aligned} bk_1 + cl_1 + \alpha bm_1 &\equiv r, \\ ck_1 \qquad + \beta bm_1 &\equiv s, \\ bl_1 + \ cm_1 &\equiv t, \pmod{p}. \end{aligned}$$

We avoid the solution of this system by taking account of the fact that if the commutator of U_3' and s_1' is expressible in terms of the commutators of U_1' and s_1' and s_2', as it must be if the resulting cubic takes the form (3), it must be independent of s_3, and hence

$$l_1 + am_1 \equiv 0, \pmod{p}.$$

Using this congruence with the last two of the system above, we get

$$\begin{aligned} k_1 &= [a^2\rho - \beta b(c - ab)(2c - ab)]/(c^2\rho), \\ l_1 &= -a(c - ab)/\rho, \\ m_1 &= (c - ab)/\rho. \end{aligned} \tag{6}$$

The first congruence of the system above then takes the form of a condition on a, b, and c. It becomes

$$3ac^2 - \alpha ab^2 + 3\beta b^2 - 2\alpha bc \equiv 0, \pmod{p}. \tag{7}$$

Hence if a, b, and c are chosen to satisfy (7), k, l, and m are determined by (5) and k_1, l_1, and m_1 by (6) so that the new generators of G satisfy (2) and the corresponding cubic, which must still be irreducible, is in the form (3). The condition that U_1', U_2', and U_3' be independent is that the determinant of the exponents of U_1, U_2, and U_3 in the expressions for them be not zero. This determinant is $(c-ab)^3/\rho^2$ and the condition is satisfied.

If we denote the transformed cubic by

$$x^3 - \alpha' x + \beta' \equiv 0, \tag{8}$$

and examine the exponents of s_3 and s_4 in the commutator of U_3' and s_1', we obtain the following congruences:

$$\begin{aligned} k_1 + al_1 + \alpha m_1 &\equiv \alpha' + b\beta', \\ ak_1 \qquad + \beta m_1 &\equiv a\alpha' + c\beta', \end{aligned}$$

which we may solve for α' and β'. The result is

$$\begin{aligned} \alpha' &= (2ab - 3\beta b + \alpha c - 3a^2 c)/\rho, \\ \beta' &= (a^3 - \alpha a + \beta)/\rho. \end{aligned} \tag{9}$$

We next take the irreducible congruence (3) and consider the effect of a linear fractional transformation on the variable x. Let this transformation be

$$x' = (-x + a)/(bx - c). \tag{10}$$

If the coefficients a, b, and c are residues (mod p) and if they satisfy (7), the transformation changes (3) into

$$x'^3 - \alpha' x' + \beta' \equiv 0,$$

where β' and α' are given by (9). In fact the condition (7) is the condition that the coefficient of x'^2 in the transformed congruence be zero. Hence the transformation (10) on the variable x is the same as the transformation (4) on the generators of G. We may then study all transformations (4) by studying transformations (10).

Now the irreducible congruence (3) defines a Galois field $GF(p^3)$ in which the congruence has three roots. Any other irreducible cubic congruence, for example (8), defines a $GF(p^3)$ simply isomorphic with the first. The congruence (8) is therefore reducible in the $GF(p^3)$ defined by (3). If X is one of the roots of (8), then the three roots are*

$$X,\ X^p, \text{ and } X^{p^2}.$$

Any mark X in the $GF(p^3)$ may be written

$$\gamma x^2 + \delta x + \epsilon, \tag{11}$$

where x is a root of (3) and γ, δ, and ϵ are rational, that is, in the $GF(p)$. Then in order to prove our theorem we need only to show that rational numbers a, b, and c can be found such that (10)

* Dickson, *Linear Groups*, p. 20.

transforms a root of (3) into (11). Setting (11) equal to the right-hand side of (10), we obtain

$$a = (\beta\gamma^2 + \delta\epsilon)/(\alpha\gamma^2 - \delta^2 + \gamma\epsilon),$$

$$b = \gamma/(\alpha\gamma^2 - \delta^2 + \gamma\epsilon), \tag{12}$$

$$c = \delta/(\alpha\gamma^2 - \delta^2 + \gamma\epsilon).$$

This shows that (4) may be chosen so that (3) is transformed into (8), where α' and β' are any numbers such that (8) is irreducible, and hence completes the proof of Theorem 1.

2. *A Special Transformation.* Much interesting information about irreducible cubics is obtained by considering special transformations of the form (4). Let $s_1' = s_1^k$, $U_2' = U_2^k$, $U_3' = U_3^{k^2}$.* This leaves G invariant and the transform of the congruence (3) corresponding to the generators s_1', s_2, U_1, U_2', U_3' is

$$x^3 - \alpha k^2 x + \beta k^3 \equiv 0. \tag{13}$$

By means of this transformation we are able to classify all cubic congruences of the form (3) in three types: those for which (a) $\alpha = 0$; (b) $\alpha = 1$; and (c) α is a particular number not a square, (mod p).

Let us consider the possibility of transforming (3) into an irreducible congruence of the same form except that $\alpha = 0$. By means of (7) we may change the value of α' in (9) to

$$\alpha' = (c - ab)[\alpha(c + ab) - 3\beta b]/(c\rho). \tag{14}$$

Hence if α' is to be zero, then a, b, and c must be chosen so that

$$[\alpha(c + ab) - 3\beta b] \equiv 0.$$

Using this with (7) to eliminate b we obtain

$$c(3\alpha a^2 - 9\beta a + \alpha^2) \equiv 0.$$

If $c = 0$ and (7) is satisfied, we must have $a = 3\beta/\alpha$ and excepting the case where $\alpha = 2$ these values require b to be zero if α' is to be zero. Therefore, in general, we must have

$$3\alpha a^2 - 9\beta a + \alpha^2 \equiv 0.$$

Since a must be rational it follows that

* This is a special case of a transformation which only *appears* to be more general than (4).

$$81\beta^2 - 12\alpha^3 \tag{15}$$

is a square. Conversely if (15) is a square, a, b, and c may be found such that α' is zero, in which case the irreducible congruence takes the form

$$x^3 + \beta' \equiv 0. \tag{16}$$

Irreducible congruences of the form (16) exist when p is of the form $6k+1$ and do not exist when p is of the form $6k-1$. The irreducibility of the congruence (3) says that the group $\{s_1, s_2\}$ contains no operator permutable with any operator of $\{U_1, U_2, U_3\}$, a property independent of the choice of generators of the two groups. Hence the irreducibility of (3) implies the irreducibility of (16). Therefore (15) is a square when p is of the form $6k+1$ and is not a square when p is of the form $6k-1$. The quantity (15) is the product of -3 and the discriminant of (3). Now -3 is or is not a square according as p is of the form $6k+1$ or $6k-1$, and consequently the discriminant of an irreducible cubic is always a square.*

3. *Method of Writing all Irreducible Cubics of the Form* (3). The following considerations give an easy method of writing all the irreducible cubics of the form (3) when p is of the form $6k+1$. Start with any irreducible cubic in the form (16). Since $\alpha=0$, the transformation (4) will have its coefficients subjected to (7) changed to

$$ac^2 + \beta b^2 \equiv 0.$$

The first relation of (9), or (14), becomes

$$\alpha' = 3ac(c - ab)/(b\rho).$$

Setting $\alpha'=1$ and combining these two relations to eliminate β, we have

$$c(c - ab)(3a - bc) \equiv 0.$$

Therefore if $\alpha'=1$, we have $3a=bc$ and we may solve for a and b in terms of c. This gives

$$a = -c^4/(9\beta), \quad b = -c^3/(3\beta). \tag{17}$$

The determinant $c-ab$ could be zero only if $c(27\beta^2-c^6)$ were zero which is impossible since β is not a cube. Hence c may be

* See Dickson, *Criteria for the irreducibility of functions in a finite field*, this Bulletin, vol. 13 (1906–7), p. 1.

chosen at random (different from zero) and a and b may then be determined so that $\alpha' = 1$. The value of β' is then

$$\beta' = (a^3 + \beta)/\rho = (27\beta^2 + c^6)/(27\beta c^3). \tag{18}$$

The condition that two values of β' given by (18) for two numbers c and c' be the same is

$$27\beta^2(c'^3 - c^3) = c'^3c^3(c'^3 - c^3).$$

Again recalling that β is not a cube, we see that this requires that $c^3 = c'^3$. Therefore, by taking $(p-1)/3$ numbers c whose cubes are distinct, we obtain $(p-1)/3$ distinct numbers β', each of which determines an irreducible cubic of the form $x^3 - x + \beta' \equiv 0$. By means of this transformation and the one which changes (3) into (13), we may obtain all the irreducible cubics (3) for which α is a square from any irreducible cubic of the form (16).

If we let α' be k, instead of 1 as above, and carry through a computation similar to the above, we obtain

$$\beta' = (27\beta^2 + c^6k^3)/(27\beta c^3). \tag{19}$$

Hence we have the following theorem.

THEOREM 2. *The irreducible cubics of the form* (3), *when* p *is of the form* $6k+1$, *are*

$$x^3 - \alpha x + (27\beta^2 + c^6\alpha^3)/(27\beta c^3) \equiv 0,$$

where $x^3 + \beta \equiv 0$ *is any irreducible binomial cubic and* c *and* α *are any numbers from* 1 *to* $p-1$.

This theorem gives a straightforward method of writing without duplications all the irreducible cubics of the form (3) when p is of the form $6k+1$ and a non-cube β is known. In the case of p of the form $6k-1$ we have not been able to obtain a like result. There is, however, one interesting formula which we shall give.

We have given an arbitrary irreducible cubic (3) and we apply the transformation (4) whose exponents are subject to the condition (7). Since a, b, and c are rational, (7) imposes a restriction on a. Also, (7) determines the ratio of c and b in terms of a. If we let $a = 3\beta/\alpha$, we find

$$\alpha' = \frac{81\beta^2}{b^2(4\alpha^3 - 27\beta^2)}. \tag{20}$$

Since the discriminant of the cubic is a square, b can be chosen so that $\alpha' = 1$. The corresponding value of β' is

$$\beta' = \frac{-729\beta^4}{\alpha^3 b^3(4\alpha^3 - 27\beta^2)}.$$

The ratio of β' to α' is $-9\beta^2/(\alpha^3 b)$. If we take α in (3) to be 1 and determine b in (20) so that α' is 1, we have

$$\beta' = -\beta(4 - 27\beta^2)^{1/2}.$$

We have therefore the following theorem.

THEOREM 3. *If p is of the form $6k-1$ and $x^3-x+\beta\equiv 0$ is irreducible, then $x^3-x+\beta(4-27\beta^2)^{1/2}\equiv 0$ is also irreducible.*

If β in the above theorem is not $1/3$, the second cubic is distinct from the first. By repeated applications of the theorem we obtain a set of cubics of the form $x^3-x+\beta\equiv 0$, but in general we do not obtain all of them.

THE UNIVERSITY OF ILLINOIS

THE ALGEBRA OF SELF-ADJOINT BOUNDARY-VALUE PROBLEMS*

BY V. V. LATSHAW

1. *Introduction.* By algebraic processes, D. Jackson† obtained in matrix form the condition for self-adjointness of differential systems of any order. The purpose of this paper is to develop by means of the matrix criterion the explicit conditions for self-adjointness of the boundary conditions associated with self-adjoint and anti-self-adjoint differential equations.

2. *Even-Order Systems.* Let $L(u)$ denote the self-adjoint differential expression‡

$$(1) \qquad L(u) \equiv (p_m u^{(m)})^{(m)} + (p_{m-1}u^{(m-1)})^{(m-1)} + \cdots + p_0 u,$$

where m is any positive integer, $p_i(x)$ is of class C^i, and $p_m(x) \neq 0$ in the interval $(a \leqq x \leqq b)$. Along with

* Presented to the Society, October 31, 1931.

† D. Jackson, Transactions of this Society, vol. 17 (1916), pp. 418–424.

‡ Bounitzky, Journal de Mathématiques, (6), vol. 5 (1909), p. 107.

$$L(u) = 0 \tag{2}$$

is given a set of $2m$ linearly independent boundary conditions:

$$U_k(u) = \sum_{i=1}^{2m} \left\{ a_{ki} u^{(i-1)}(a) + b_{ki} u^{(i-1)}(b) \right\} = 0.$$

The theorem of Jackson is as follows.

*If the differential expression $L(u)$ is self-adjoint, the condition that the boundary conditions be self-adjoint is that the matrix**

$$\boldsymbol{\delta}' \boldsymbol{\pi}^{(1)} \boldsymbol{\delta} - 2\boldsymbol{\pi}^{(4)} \boldsymbol{\delta} + \boldsymbol{\pi}^{(2)}$$

be symmetric; if $L(u)$ is an anti-self-adjoint expression or differs from such an expression only in the terms of order zero, the condition is that the matrix just written down be skew symmetric.

If we denote the elements of the non-singular matrix $\boldsymbol{A}_1$ by q_{ij} and those of $\boldsymbol{A}_2$ by $\dot{q}_{ij}$, we have

$$\boldsymbol{A}_1^{-1} = (Q1, 1; 2m, 2m)'/\overline{A}_1,$$

where Q_{ij} represents the cofactor of q_{ij} in $\overline{A}_1$. Since $\boldsymbol{\pi}^{(1)}$ is skew symmetric, we find that the element in the ith row and jth column of the matrix $\boldsymbol{\delta}' \boldsymbol{\pi}^{(1)} \boldsymbol{\delta}$ may be written

$$\frac{1}{\overline{A}_1^2} \sum_{r=1}^{2m-1} \sum_{s=r+1}^{2m} \pi_{rs}^{(1)} \left[\sum_{n,k=1}^{2m} \dot{q}_{ni} \dot{q}_{kj} (Q_{nr} Q_{ks} - Q_{ns} Q_{kr}) \right].$$

The summation over the indices n and k is equivalent to

$$\sum_{n=1}^{2m-1} \sum_{k=n+1}^{2m} \left[(\dot{q}_{ni} \dot{q}_{kj} - \dot{q}_{nj} \dot{q}_{ki})(Q_{nr} Q_{ks} - Q_{ns} Q_{kr}) \right].$$

---

* We shall adopt the following notation:

$$(ap, q; r, s) = \begin{pmatrix} a_{pq} \cdots a_{ps} \\ \cdots \\ a_{rq} \cdots a_{rs} \end{pmatrix}; \qquad (ap, q; r, s)' = \begin{pmatrix} a_{pq} \cdots a_{rq} \\ \cdots \\ a_{ps} \cdots a_{rs} \end{pmatrix}.$$

Boldface letters denote matrices and dashed italics determinants. If we let $\boldsymbol{A}_1$ be a square non-singular matrix of the coefficients of $2m$ variables in $U_k(u)$ and $\boldsymbol{A}_2$ the matrix of the remaining $2m$ variables, then $\boldsymbol{\delta}$ denotes the product $\boldsymbol{A}_1^{-1} \boldsymbol{A}_2$ and $\boldsymbol{\delta}'$ the conjugate of $\boldsymbol{\delta}$. Further, we have from Green's formula,

$$\int_a^b [vL(u) - uL(v)]dx = \pi(u, v),$$

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} & \boldsymbol{\pi}^{(3)} \\ \boldsymbol{\pi}^{(4)} & \boldsymbol{\pi}^{(2)} \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\pi}^{(1)} 1, 1; 2m, 2m)(\boldsymbol{\pi}^{(3)} 1, 1; 2m, 2m) \\ (\boldsymbol{\pi}^{(4)} 1, 1; 2m, 2m)(\boldsymbol{\pi}^{(2)} 1, 1; 2m, 2m) \end{pmatrix}.$$

We replace the quantity $Q_{nr}Q_{ks}-Q_{ns}Q_{kr}$ by $\overline{A}_1$ [alg. comp. of $(q_{nr}q_{ks}-q_{ns}q_{kr})$] and obtain the further equivalent expression

$$\overline{A}_1\Big[\sum_{n=1}^{2m-1}\sum_{k=n+1}^{2m}(\dot{q}_{ni}\dot{q}_{kj}-\dot{q}_{nj}\dot{q}_{ki})\{\text{alg. comp. of } (q_{nr}q_{ks}-q_{ns}q_{kr})\}\Big].$$

The expansion within the brackets is Laplace's development of the determinant $\overline{A}_1$ with the rth and sth columns replaced by the ith and jth columns, respectively, of A_2. This determinant will be denoted by $D_{rs}(i, j)$.

The application of the criterion of the theorem cited, which required the product of five matrices, is reduced to a process summarized in the following theorem.

THEOREM 1. *Given the self-adjoint differential equation, $L(u)=0$, defined by (1), where the coefficients are of class C^i, and $p_m(x)\neq 0$ in the interval $(a\leqq x\leqq b)$, and given a set of $2m$ linear boundary conditions*

$$U_k(u)=\sum_{i=1}^{2m}\{a_{ki}u^{(i-1)}(a)+b_{ki}u^{(i-1)}(b)\}=0,$$

where the determinant $\overline{A}_1=|q_{ij}|$ is of rank $2m$; then the condition for self-adjointness of the system thus defined, that is, that the matrix

$$(A_1^{-1}A_2)'\pi^{(1)}A_1^{-1}A_2-2\pi^{(4)}A_1^{-1}A_2+\pi^{(2)}$$

be symmetric, reduces by the removal of the factor A_1 to the following $m(2m-1)$ conditions:

$$T_{ij}=\sum_{r=1}^{2m-1}\sum_{s=r+1}^{2m}\pi_{rs}^{(1)}D_{rs}(i,j)-\sum_{r=1}^{2m}\pi_{ir}^{(4)}D_r(j)+\sum_{r=1}^{2m}\pi_{jr}^{(4)}D_r(i)+\overline{A}_1\pi_{ij}^{(2)}=0,\qquad (i<j\leqq 2m).$$

The symbol $D_{rs}(i, j)$ denotes the determinant A_1 with the rth and sth columns replaced by the ith and jth columns, respectively, of A_2; $D_r(i)$ indicates one replacement only.

We illustrate the theorem with an example. Let us consider the second-order system

$$(p_1u')' + p_0u = 0, \quad \sum_{i=1}^{2}\left\{a_{ki}u^{(i-1)}(a) + b_{ki}u^{(i-1)}(b)\right\} = 0, \qquad (k = 1, 2).$$

According to the theorem there is one condition for self-adjointness of the boundary conditions, namely,

$$T_{12} = \pi_{12}^{(1)}D_{12}(1, 2) - \pi_{11}^{(4)}D_1(2) - \pi_{12}^{(4)}D_2(2) + \pi_{21}^{(4)}D_1(1) + \pi_{22}^{(4)}D_2(1) + \overline{A}_1\pi_{12}^{(2)} = 0.$$

For the sake of definiteness let us assume the non-singular matrix to be

$$A_1 = \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix}; \qquad \text{then} \qquad A_2 = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix}.$$

Accordingly we have

$$\pi^{(1)} = 0, \qquad \pi^{(2)} = 0,$$

$$\pi^{(3)} = \begin{pmatrix} -p_1(a) & 0 \\ 0 & p_1(b) \end{pmatrix}, \qquad \pi^{(4)} = \begin{pmatrix} p_1(a) & 0 \\ 0 & -p_1(b) \end{pmatrix},$$

$$T_{12} = p_1(a)\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} - p_1(b)\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

3. *Special Cases.* In the explicit characterization of specific cases we assume first that in the $2m$ boundary conditions the sets of terms involving the point a are linearly independent. Under this assumption the condition for self-adjointness becomes

$$(A_1^{-1}A_2)'\pi^{(1)}A_1^{-1}A_2 + \pi^{(2)} = 0,$$

subject to the following definitions:

$$(3) \qquad A_1 = (a1, 1; 2m, 2m), \qquad A_2 = (b1, 1; 2m, 2m),$$

$$\pi = \begin{pmatrix} \pi^{(1)} & \pi^{(3)} \\ \pi^{(4)} & \pi^{(2)} \end{pmatrix} = \begin{pmatrix} (\alpha 1, 1; 2m, 2m) & 0 \\ 0 & (\beta 1, 1; 2m, 2m) \end{pmatrix}^{*}.$$

* The quantities α_{rs} and β_{rs} are given by the relations

$$\alpha_{rs} = (-1)^{r+1}G(a), \quad \beta_{rs} = (-1)^{r}G(b), \text{ where } G(x) = \sum_{n}\binom{n-r}{k}p_n^{(k)}(x),$$

if $(r \leqq n \leqq m, s-1)$, $(k = 2n + 1 - r - s \geqq 0)$, and $r < s$. Further we set

$$\binom{n-r}{0} = 1.$$

We incorporate these results in a corollary.

COROLLARY. *If in Theorem* 1 *we assume that the determinants* $\overline{A}_1 = |a_{ij}|$ *and* $\overline{A}_2 = |b_{ij}|$ *are of rank* $2m$, *then the condition for self-adjointness of the system thus defined reduces to the following* $m(2m-1)$ *explicit conditions:*

$$T_{ij} = \sum_{r=1}^{m} \sum_{s=r+1}^{2m} \alpha_{rs} D_{rs}(i, j) + \overline{A}_1 \beta_{ij} = 0, \qquad (i < j \leqq 2m).$$

The quantities α_{rs} *and* β_{rs} *are computed from the relations* (3).

An example will illustrate the corollary. Let us consider the second-order system

$$(p_1 u')' + p_0 u = 0, \quad \sum_{i=1}^{2} \{a_{ki} u^{(i-1)}(a) + b_{ki} u^{(i-1)}(b)\} = 0.$$

One explicit condition for self-adjointness is obtained:

$$T_{12} = \alpha_{12} D_{12}(1, 2) + \overline{A}_1 \beta_{12} = 0.$$

From (3) we have $\alpha_{12} = p_1(a)$ and $\beta_{12} = -p_1(b)$; also we observe

$$D_{12} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \quad \text{and} \quad \overline{A}_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

We now write the condition in the well known form

$$T_{12} = p_1(a) \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} - p_1(b) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

As a second special case let us consider a system consisting of the self-adjoint differential equation defined in (2) and a set of $2m$ linearly independent Sturmian boundary conditions

$$U_k^{(a)} = \sum_{i=1}^{2m} a_{ki} u^{(i-1)}(a) = 0, \qquad (k = 1, 2, \cdots, h),$$

$$U_k^{(b)} = \sum_{i=1}^{2m} b_{ki} u^{(i-1)}(b) = 0, \qquad (k = 1, 2, \cdots, f).$$

It can be shown* that if h of the given boundary conditions

* D. Jackson, Proceedings of the American Academy of Arts and Sciences, vol. 51, pp. 403–417.

involve only the point a, then h of the adjoint conditions will involve only the point b; consequently for $h \neq f$ self-adjointness is impossible. We shall assume $h = f = m$ and that the sets of terms involving the variables $u(a), u'(a), \cdots, u^{(m-1)}(a)$ and likewise those involving the variables $u(b), u'(b), \cdots, u^{(m-1)}(b)$ are linearly independent. Following the process previously outlined, we find

$$(4)\qquad A_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} (a1, 1; m, m) & 0 \\ 0 & (b1, 1; m, m) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} (a1, m+1; m, 2m) & 0 \\ 0 & (b1, m+1; m, 2m) \end{pmatrix}.$$

Further, let $D_r(i)$ represent the determinant $\bar{a}$ with elements a_{kr} replaced by elements a_{ki}. Also let $D_{rs}(i, j)$ denote $\bar{a}$ with a_{kr} and a_{ks} replaced by a_{ki} and a_{kj} respectively. Similarly, we represent the determinants involving b by $D_r'(i)$ and $D_{rs}'(i, j)$.

After making certain algebraic reductions, we are able to express the product matrices $\delta' \pi^{(1)} \delta$ and $\delta \pi^{(4)}$ in concise form:

$$\delta' \pi^{(1)} \delta = \begin{pmatrix} (a'1, 1; m, m) & 0 \\ 0 & (b'1, 1; m, m) \end{pmatrix},$$

$$\pi^{(4)} \delta = \begin{pmatrix} (a''1, 1; m, m) & 0 \\ 0 & (b''1, 1; m, m) \end{pmatrix},$$

where

$$a_{ij}' = \frac{1}{\bar{a}} \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} \alpha_{rs} D_{rs}(m+i, m+j),$$

$$b_{ij}' = \frac{1}{\bar{b}} \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} \beta_{rs} D_{rs}'(m+i, m+j),$$

and

$$a_{ij}'' = \frac{1}{\bar{a}} \sum_{s=1}^{m} \alpha_{m+j,s} D_s(m+i), \quad b_{ij}'' = \frac{1}{\bar{b}} \sum_{s=1}^{m} \beta_{m+j,s} D_s'(m+i).$$

We summarize these results in a theorem.

THEOREM 2. *Given the self-adjoint differential equation* $L(u) = 0$, *defined by* (1), *where the coefficients are of class* C^i *and we assume* $p_m(x) \neq 0$ *in the interval* $(a \leqq x \leqq b)$; *and given a set of* $2m$ *linearly independent Sturmian boundary conditions*:

$$U_k(a) = \sum_{i=1}^{2m} a_{ki}u^{(i-1)}(a) = 0, \quad U_k(b) = \sum_{i=1}^{2m} b_{ki}u^{(i-1)}(b) = 0,$$
$$(k = 1, 2, \cdots, m),$$

in which $|a_{ks}|, |b_{ks}| \neq 0$ *for* $s = 1, 2, \cdots, m$; *the condition for self-adjointness of the system thus defined, that is, that the matrix*

$$(A_1^{-1}A_2)'\pi^{(1)}A_1^{-1}A_2 - 2\pi^{(4)}A_1^{-1}A_2 + \pi^{(2)}$$

be symmetric, reduces by the removal of common factors to the following $m(m-1)/2$ *conditions at each end point:*

$$T_{ij}(a) = \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} \alpha_{rs}D_{rs}(m+i, m+j) + \sum_{s=1}^{m} \alpha_{m+j,s}D_s(m+i) - \sum_{s=1}^{m} \alpha_{m+i,s}D_s(m+j) = 0,$$

$$T_{ij}(b) = \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} \beta_{rs}D_{rs}'(m+i, m+j) + \sum_{s=1}^{m} \beta_{m+j,s}D_s'(m+i) - \sum_{s=1}^{m} \beta_{m+i,s}D_s'(m+j) = 0, \qquad (i < j \leqq m),$$

in which the abbreviations defined in (3) *and* (4) *are used.*

EXAMPLE 1. $m = 1$. According to the requirements of Theorem 2 this system is always self-adjoint. It is of interest to check this result by direct calculation.

$$\int_a^b [vL(u) - uL(v)]dx = [p_1(x)(vu' - uv')]_a^b = U_1V_4 + U_2V_3 + U_3V_2 + U_4V_1,$$

where $U_1 = a_{11}u(a) + a_{12}u'(a)$, $U_2 = b_{11}u(b) + b_{12}u'(b)$. Further we assume $U_3 = u'(a)$, $U_4 = u'(b)$; then

$$V_1 = p_1(b)[b_{11}v(b) + b_{12}v'(b)]/b_{11},$$

$$V_2 = -p_1(a)[a_{11}v(a) + a_{12}v'(a)]/a_{11}.$$

Since V_1 and V_2 are essentially U_2 and U_1, respectively, the system is self-adjoint.

EXAMPLE 2. $m=2$.

$$T_{12}(a) = p_1(a)\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - p_2(a)\begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} + p_2'(a)\begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} - p_2(a)\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} = 0,$$

$$-T_{12}(b) = p_1(b)\begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix} - p_2(b)\begin{vmatrix} b_{13} & b_{12} \\ b_{23} & b_{22} \end{vmatrix} + p_2'(b)\begin{vmatrix} b_{14} & b_{12} \\ b_{24} & b_{22} \end{vmatrix} - p_2(b)\begin{vmatrix} b_{11} & b_{14} \\ b_{21} & b_{24} \end{vmatrix} = 0.$$

4. *Odd-Order Systems.* In general we shall use the same method but some modification is required. Let $L(u)$ represent the anti-self-adjoint differential expression

$$(5)\quad L(u) \equiv (p_n u)^{(n)} + (p_{n-2}u)^{(n-2)} + \cdots + (p_1 u)' + p_n u^{(n)} + p_{n-2}u^{(n-2)} + \cdots + p_1 u',$$

where n is any positive odd integer, $p_i(x)$ is of class C^i, and $p_n(x) \neq 0$ in the interval $(a \leqq x \leqq b)$. Along with

$$L(u) = 0$$

is given a set of linearly independent boundary conditions

$$U_k = \sum_{i=1}^{n} \{ a_{ki}u^{(i-1)}(a) + b_{ki}u^{(i-1)}(b) \} = 0, \quad (k = 1, 2, \cdots, n).$$

In our usual notation the product matrix $\delta'\pi^{(1)}\delta$ may be written

$$\delta'\pi^{(1)}\delta = \frac{1}{A_1^2}(a'1, 1; n, n), \text{ where } a_{ij}' = \sum_{r,s=1}^{n} \pi_{rs}^{(1)} D_r(i) D_s(j).$$

Since $\pi^{(1)}$ is not skew symmetric for the odd-order system, the further reduction by removal of a common factor is no longer possible. We state these results more concisely in the following theorem.

THEOREM 3. *Given the anti-self-adjoint differential equation* $L(u)=0$, *defined by* (5), *where the coefficients* $p_i(x)$ *are of class* C^i, *and* $p_n(x) \neq 0$ *in the interval* $(a \leqq x \leqq b)$, *and given a set of n linear boundary conditions*

$$U_k = \sum_{i=1}^{n} \{ a_{ki} u^{(i-1)}(a) + b_{ki} u^{(i-1)}(b) \} = 0, \quad (k = 1, 2, \cdots, n),$$

where the determinant $\overline{A}_1 = |q_{ij}|$ *is of rank* n*; the condition for self-adjointness of the system thus defined, that is, that the matrix*

$$(\boldsymbol{A}_1^{-1}\boldsymbol{A}_2)'\boldsymbol{\pi}^{(1)}\boldsymbol{A}_1^{-1}\boldsymbol{A}_2 - 2\boldsymbol{\pi}^{(4)}\boldsymbol{A}_1^{-1}\boldsymbol{A}_2 + \boldsymbol{\pi}^{(2)}$$

be skew symmetric, reduces to the following $n(n+1)/2$ *conditions*:

$$T_{ij} = \sum_{r,s=1}^{n} \pi_{rs}^{(1)} D_r(i) D_s(j) - \overline{A}_1 \sum_{r=1}^{n} \pi_{ir}^{(4)} D_r(j) - \overline{A}_1 \sum_{r=1}^{n} \pi_{jr}^{(4)} D_r(i) + \overline{A}_1^2 \pi_{ij}^{(2)} = 0, \quad (i \leqq j \leqq n).$$

To illustrate the theorem let us consider the first-order system

$$(p_1 u)' + p_1 u' = 0, \quad a_{11} u(a) + b_{11} u(b) = 0.$$

Assume $A_1 = b_{11}$; then we have $A_2 = a_{11}$, and

$$\pi_{11}^{(1)} D_1^2(1) + A_1^2 \pi_{11}^{(2)} = 0 = a_{11}^2 p_1(b) - b_{11}^2 p_1(a).$$

5. *Special Cases.* If we assume that in the n boundary conditions the sets of terms involving the point a are linearly independent, the condition for self-adjointness becomes

$$(\boldsymbol{A}_1^{-1}\boldsymbol{A}_2)'\boldsymbol{\pi}^{(1)}\boldsymbol{A}_1^{-1}\boldsymbol{A}_2 + \boldsymbol{\pi}^{(2)} = 0,$$

subject to the following definitions:*

$$(6) \qquad \boldsymbol{A}_1 = (a1, 1; n, n), \qquad \boldsymbol{A}_2 = (b1, 1; n, n),$$

$$\boldsymbol{\pi} = \begin{pmatrix} (\alpha 1, 1; n, n) & 0 \\ 0 & (\beta 1, 1; n, n) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\pi}^{(1)} & \boldsymbol{\pi}^{(3)} \\ \boldsymbol{\pi}^{(4)} & \boldsymbol{\pi}^{(2)} \end{pmatrix}.$$

If we let $D_r(i)$ denote the determinant $\overline{A}_1$ with the elements a_{kr} replaced by the elements b_{ki}, the product matrix $\boldsymbol{\delta}'\boldsymbol{\pi}^{(1)}\boldsymbol{\delta}$ may be written

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* $\alpha_{rs} = \alpha_{sr} = \sum_m G_m p_{2m+1}^{(k)}(a)$; $\beta_{rs} = \beta_{sr} = -\sum_m G_m p_{2m+1}^{(k)}(b)$, where

$$G_m = (-1)^s \binom{2m-s+1}{k} + (-1)^r \binom{2m-r+1}{k},$$

if $k = 2m-r-s+2 \geqq 0$ and $0 \leqq m \leqq (n-1)/2$.

$$\delta'\pi^{(1)}\delta = \frac{1}{\overline{A}_1^2}(a'1, 1; n, n), \text{ where } a_{ij}' = \sum_{r,s=1}^{n} \alpha_{rs}D_r(i)D_s(j).$$

These results may be stated in a corollary.

COROLLARY. *If in Theorem 3 we assume that the determinants* $\overline{A}_1 = |a_{ij}|$ *and* $\overline{A}_2 = |b_{ij}|$ *are of rank* n, *then the condition for self-adjointness of the system thus defined reduces to the following* $n(n+1)/2$ *explicit conditions*:

$$T_{ij} = \sum_{r,s=1}^{n} \alpha_{rs}D_r(i)D_s(j) + \overline{A}_1^2\beta_{ij} = 0, \qquad (i \leqq j \leqq n).$$

The quantities α_{rs} *and* β_{rs} *are obtained from the relations* (6).

An example will suffice to illustrate the corollary. For $n=1$,

$$(p_1u)' + p_1u' = 0, \quad a_{11}u(a) + b_{11}u(b) = 0.$$

One condition for self-adjointness is specified, namely:

$$T_{11} = p_1(b)a_{11}^2 - p_1(a)b_{11}^2 = 0.$$

In verifying this result by direct calculation we have

$$\pi(u, v) = 2p_1(b)v(b)u(b) - 2p_1(a)v(a)u(a) = U_1V_2 + U_2V_1,$$

where $U_1 = a_{11}u(a) + b_{11}u(b)$ and we let $U_2 = u(b)$. We find

$$V_1 = 2p_1(b)v(b) + 2p_1(a)v(a)b_{11}/a_{11}.$$

Obviously for V_1 to be essentially U_1 the condition stated above must hold.

In considering odd-order systems with Sturmian boundary conditions it is sufficient to observe that the conditions pertaining to the end points are unequal in number; consequently self-adjointness of such systems is impossible.

LEHIGH UNIVERSITY

KOJIMA ON DOUBLE DIRICHLET SERIES

BY C. R. ADAMS

The object of this note is to direct attention to the most extensive and for us one of the most interesting papers by Tetsuzô Kojima, whose death on April 7, 1921, at the early age of thirty-four deprived Japan of an able and promising mathematician. This paper, which has only just now come to our notice* and whose existence may not be generally known, is *On the double Dirichlet series*, (Science Reports of the Tôhoku Imperial University, (1), (math., phys., chem.), vol. 9 (1920), pp. 351–400). It is devoted mainly to the convergence properties of double Dirichlet series, certain related double integrals, double factorial series, and double binomial coefficient series. Unaware of Kojima's work, Leja and the present writer† have recently considered some of the same questions; it may therefore be appropriate to give a brief description of Kojima's memoir and to point out the main features of its relationship to the more recent papers cited.

A double series $\sum_{i,j=1}^{\infty} a_{ij}$ for which the sequence $\{S_{mn}\}$ of partial sums, $S_{mn}=\sum_{i,j=1}^{m,n} a_{ij}$, converges as $m, n\to\infty$ and of which each row and column converges has been called *regularly convergent* by Hardy. If $\{S_{mn}\}$ is both convergent and bounded, the series may be said to be *convergent-bounded*. One sees at once that every series regularly convergent is convergent-bounded but that the converse is not true. The notion of regular convergence is basic in Kojima's work; that of convergence-bounded in L, AI, and AII.

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* An obituary in the Tôhoku Mathematical Journal, vol. 19 (1921), 2 pp. following the table of contents, gives a complete list of Kojima's fifteen papers, including two published posthumously, together with an account (in Japanese) of his life.

† Leja, *Sur les séries de Dirichlet doubles*, Comptes-Rendus du 1 Congrès des Mathématiciens des Pays Slaves, Warsaw, 1930, pp. 140–158; Adams, *On multiple factorial series*, Annals of Mathematics, (2), vol. 32 (1931), pp. 67–82; and Adams, *Note on multiple Dirichlet and multiple factorial series*, ibid., vol. 33 (1932), pp. 406–412. Hereafter these papers will be referred to respectively as L, AI, and AII.

Kojima begins (Theorem I and ensuing discussion) by establishing the existence of related abscissas of regular convergence S, T for the Dirichlet series

$$(1) \qquad \sum_{i,j=1}^{\infty} a_{ij} e^{-\lambda_i s - \mu_j t},$$

$\{\lambda_i\}$ being a sequence of real numbers monotonically increasing without limit and $\{\mu_j\}$ a like sequence. The existence of related abscissas of convergence-bounded is an immediate consequence of Theorem 1, part 1, of L, which is the same as Theorem 1 of AII. That a pair of abscissas of regular convergence is also a pair of abscissas of convergence-bounded, and conversely, follows at once from remarks in both L and AII. The precise significance of having one abscissa finite and its related abscissa infinite is specified only by Kojima.

In Theorem II, Kojima gives for double series a generalization of his own formula for the convergence abscissa, *not necessarily positive*, of a simple Dirichlet series.* Theorem III, stating a necessary and sufficient condition that a pair of numbers S, T, neither necessarily positive, be related abscissas of regular convergence, follows easily. A similar investigation of pairs of related *positive* abscissas, suggested by the Cahen formula for the convergence abscissa (positive) of a simple Dirichlet series, is made in L, the results being contained in Theorems III, IV, and V. Regarding T as a function of S, Kojima then proves the convexity of this function and subjects it to a careful further analysis. This is paralleled, in part, by Theorem 6 of L and the attendant discussion.

Theorem IV is an analog of Theorem II for absolute convergence. Theorem V, obviously mis-stated, should be given the following form: *If S, T, is any pair of related abscissas of regular convergence, there always exists a pair of related abscissas of absolute convergence $\overline{S}$, $\overline{T}$ satisfying the inequalities*

$$\overline{S} - S \leqq \limsup_{m\to\infty} \frac{\log m}{\lambda_m}, \qquad \overline{T} - T \leqq \limsup_{n\to\infty} \frac{\log n}{\mu_n}.$$

Theorem VI contains a generalization of Fujiwara's formula for

* This formula, given in 1914, he had shown to be a natural extension of the Cauchy-Hadamard formula for the radius of convergence of a power series.

the convergence abscissa (not necessarily positive) of a simple Dirichlet series.

Extensions of theorems of Vitali on sequences of analytic functions are given as Theorems VII and VIII. Theorem IX, concerning uniform convergence in a finite region and the consequent analyticity of the function represented, is essentially duplicated by Theorem 1, part 2, of L, but is less general than Theorem 2 of AII on uniform convergence for an infinite region. Theorem X asserts that if the coefficients in (1) are all positive, (S, T) is a singular place for the function represented. Theorem XI, on simple Dirichlet series, is a preliminary to the proof of Theorem XII on the convergence of (1) in the region $\mathcal{R}(s) = S$, $\mathcal{R}(t) = T$.

Theorem XIII, on regular convergence of a double factorial series, is slightly less general than Theorem 1 of AI as supplemented by remarks on page 70 of AI and strengthened on page 407 of AII. Theorem XIV, on the identity of the related half-planes of regular convergence for a factorial series and its associated special Dirichlet series, is duplicated by Theorem 3 of AII. Theorem XV, on uniform convergence of a factorial series in a finite region, is slightly less general than the similar Theorem 2 of AI and much less general than Theorem 7 of AI (supplemented by footnote 12 of AII) on uniform convergence for an infinite region. Theorems XVI, XVII, and XVIII are respective analogs for binomial coefficient series of the preceding three theorems on factorial series.

BROWN UNIVERSITY

A CENSUS OF SQUARES OF ORDER 4, MAGIC IN ROWS, COLUMNS, AND DIAGONALS*

BY D. N. LEHMER

This census was made by Frenicle in 1693 who found 880 fundamental squares which with their rotations and reflections gave 7,040 squares of this sort. A study of the transformations which throw a square of this type into another of the same type shows that Frenicle need not have listed more than 220 fundamental squares. Besides his transformations, which were a rotation R of order 4 of the square in its plane about a right angle,

* Presented to the Society, December 2, 1933.

and a reflection T of order 2 in a diagonal, there is a transformation S of order 2 obtained by interchanging the 1st and 2d rows and the 1st and 2d columns. Also there is a transformation U of order 2 obtained by interchanging the two middle rows and the two middle columns. These with the product transformation US serve to associate with any magic square of the sort in question 31 others so that the total number of squares should be a multiple of 32, and if Frenicle's result were correct one need list only 220 fundamental squares.

A set of fundamental squares must be such that no one of them can be obtained from any other by any of the transformations indicated above. The defining conditions were made to depend on the interesting fact that all of these transformations interchanged the *corner elements* among themselves, and the *side elements* among themselves. By a *corner element* is meant one that occupies a corner cell of the square or one of the four central cells. By a *side element* is meant any other. Attention was centered on the two largest entries 15 and 16, and the squares were listed in four groups: (I) those in which 15 and 16 were corner elements; (II) those in which 16 was a corner element and 15 a side element; (III) those in which 15 was a corner element and 16 a side element; and (IV) those in which 15 and 16 were both side elements. Of the first set there were 31 fundamental squares; of the second there were 71; of the third there were also 71 and of the fourth there were 47. The total $31+71+71+47=220$, and Frenicle's result is checked again, the total number being $220 \cdot 32 = 7{,}040$.

From this list of fundamental squares it is easy to pick out 6 which are magic also in the broken diagonals. It is interesting to note, however, that while the transformations R, S, and T leave this property of "diabolism" unaltered, the transformations U and US destroy it. The total number of such squares is therefore $6 \cdot 8 = 48$. The complete results for all squares of order 4 are then as follows:

Number of squares magic in rows and columns..539,136*,
Number magic in rows, columns, and main diagonals...7,040,
Number magic in rows, columns, and all diagonals.....48.

THE UNIVERSITY OF CALIFORNIA

* This Bulletin, vol. 39 (1933), p. 767.

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ERRATA

This Bulletin, vol. 38, No. 12 (Dec., 1932):

Page 931, line 25: *insert* Montgomery, D., (813); *before* Moore.

This Bulletin, vol. 39, No. 1 (Jan., 1933):

Page 20, line 8: *for* E. H. McAllister *read* E. H. McAlister

This Bulletin, vol. 39, No. 9 (Sept., 1933):

Page 664, line 37: *for* K. Jessen *read* B. C. Jessen.

This Bulletin, vol. 39, No. 10 (Oct., 1933):

Page 764: omit footnote (paper not presented on date given).

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

CAMBRIDGE, MASSACHUSETTS, ANNUAL MEETING, December 27-29, 1933.

This meeting is to be held in conjunction with the A.A.A.S. and the M.A.A. There will be a symposium on *General Analysis;* the principal address will be by Professor T. H. Hildebrandt, and Professors R. W. Barnard and M. H. Stone will also present papers, by invitation. The Bôcher Prize will be awarded.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 30-31, 1934.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than March 9. Abstracts received by the Secretaries by that date will appear in the March issue of the BULLETIN. By invitation of the program committee, there will be a symposium on *Algebraic geometry,* under the leadership of Professors A. B. Coble, Solomon Lefschetz, Oscar Zariski, and Virgil Snyder.

CHICAGO, ILLINOIS, April 6-7, 1934.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wis., not later than March 9. Abstracts received by the Secretaries by that date will appear in the March issue of the BULLETIN. At this meeting, Professor R. E. Langer will give an address on *The asymptotic solution of linear ordinary differential equations.*

BERKELEY, CALIFORNIA, June 20, 1934.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than May 30. Abstracts received by the Secretaries by May 6 will appear in the May issue of the BULLETIN.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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